DEPARTMENT OF MATHEMATICS

MATH2000
Conservative vector fields and line integrals

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(1) The velocity potential for a vortex flow is $\phi = -\frac{\gamma}{2\pi}\tan^{-1}(\frac{y}{x})$. Find the velocity field $\boldsymbol{v} = \nabla\phi$.

(2) The potential of a point charge at $\boldsymbol{r}_0 = x_0\boldsymbol{i} + y_0\boldsymbol{j} + z_0\boldsymbol{k}$ is $\phi(x, y, z) = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}$. Find the electrical field $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$.

(3) The vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ is conservative. Find a corresponding potential function. That is, find a function $f(x, y, z)$ such that $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$.

(4) Evaluate $\displaystyle\int_C xdx + xydy$ where C is the line $y = 1 - x$ for $0 \le x \le 1$.

(a) By using the parameterization $x(t) = t^2$ & $y(t) = 1 - t^2$ for $0 \le t \le 1$.

(b) By using the parameterization $x(t) = \sin t$ & $y(t) = 1 - \sin t$ for $0 \le t \le \frac{\pi}{2}$

Note that one would normally choose $x(t) = t$ & $y(t) = 1-t$ to evaluate this integral, but the above choices are selected to demonstrate that the choice of parameterization does not affect the answer.

(5) The vector field $\mathbf{F} = (x + z)\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$ is conservative.

(a) Use the definition of the line integral to evaluate $\int \mathbf{F}\cdot d\mathbf{r}$ along the path $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ for $0 \le t \le 1$. (The goal here is carry out the full calculation, rather than using the fundamental theorem for line integrals).

(b) Find a corresponding potential function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

(c) Use the result calculated in (b) to re-evaluate the integral in (a).

(d) Evaluate the line integral $\int \mathbf{F}\cdot d\mathbf{r}$ along the path $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)\mathbf{j} + \sin^5 t\mathbf{k}$ for $0 \le t \le \frac{\pi}{2}$.

(6) (Stewart, Q7, p1079.) Evaluate the line integral

$$\int_C (xy\ dx + (x - y)\ dy)$$

where C consists of line segments from $(0, 0)$ to $(2, 0)$ and from $(2, 0)$ to $(3, 2)$.

(7) Evaluate the line integral $\displaystyle\int_C 3x^2y^2\ dx + 2x^3y\ dy$, where the path of integration C is along the curve $y = x^3 - 3x^2 + 2x$ from $(0, 0)$ to $(1, 0)$. (Hint: is the vector field conservative?)