DEPARTMENT OF MATHEMATICS

MATH2000
Conservative vector fields and line integrals (solutions)

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(1) Velocity $\boldsymbol{v} = \nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j}$.

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= \frac{y\gamma}{2\pi x^2}\frac{1}{1+(\frac{y}{x})^2} \\
&= \frac{\gamma y}{2\pi(x^2+y^2)} \\
\frac{\partial\phi}{\partial y} &= -\frac{\gamma}{2\pi x}\frac{1}{1+(\frac{y}{x})^2} \\
&= -\frac{\gamma x}{2\pi(x^2+y^2)} \\
\Rightarrow \boldsymbol{v} &= \frac{\gamma y}{2\pi(x^2+y^2)}\boldsymbol{i} - \frac{\gamma x}{2\pi(x^2+y^2)}\boldsymbol{j}.
\end{aligned}$$

(2) Method 1:

$$\phi(x,y,z) = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= -\frac{(x-x_0)}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{\frac{3}{2}}} \\
\frac{\partial\phi}{\partial y} &= -\frac{(y-y_0)}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{\frac{3}{2}}} \\
\frac{\partial\phi}{\partial z} &= -\frac{(z-z_0)}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{\frac{3}{2}}} \\
\Rightarrow \boldsymbol{E} &= \frac{(x-x_0)\boldsymbol{i}+(y-y_0)\boldsymbol{j}+(z-z_0)\boldsymbol{k}}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{\frac{3}{2}}}
\end{aligned}$$

Method 2:

Let $R = |\boldsymbol{r}-\boldsymbol{r}_0| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Then $\dfrac{\partial R}{\partial x} = \dfrac{x-x_0}{R}$ (see above). Now

$$\frac{\partial}{\partial x}\frac{1}{R} = \left(\frac{\partial}{\partial R}\frac{1}{R}\right)\frac{\partial R}{\partial x} = -\frac{1}{R^2}\frac{x-x_0}{R} = -\frac{x-x_0}{R^3}$$

so

$$\boldsymbol{E} = \left(-\frac{x-x_0}{R^3}, -\frac{y-y_0}{R^3}, -\frac{z-z_0}{R^3}\right).$$

(3) Since $\mathbf{F}(x,y,z)$ is conservative, there is a function $f(x,y,z)$ such that

$$\begin{aligned}\mathbf{F}(x,y,z) &= \bigtriangledown f(x,y,z) \\ &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= 2x\mathbf{i} + z\mathbf{j} + y\mathbf{k}. \qquad (1)\end{aligned}$$

Hence $\frac{\partial f}{\partial x} = 2x$ which implies

$$f(x,y,z) = x^2 + g(y,z). \qquad (2)$$

From (2), $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$, but from (1), $\frac{\partial f}{\partial y} = z$. Hence

$$\begin{aligned}\frac{\partial g}{\partial y} &= z \\ \Rightarrow g(y,z) &= yz + h(z) \\ \Rightarrow f(x,y,z) &= x^2 + yz + h(z). \qquad (3)\end{aligned}$$

From (1), $\frac{\partial f}{\partial z} = y$ and from (3), $\frac{\partial f}{\partial z} = y + \frac{dh}{dz}$. Hence $\frac{dh}{dz} = 0 \Rightarrow h(z) = c$, where c is a constant. Thus $f(x,y,z) = x^2 + yz + c$.

(4) By definition $I = \int_C xdx + xydy = \int_a^b x(t)x'(t)dt + x(t)y(t)y'(t)dt$ for $a \le t \le b$.

(a) $x'(t) = 2t$ and $y'(t) = -2t$ so

$$I = \int_0^1 t^2 2tdt + t^2(1-t^2)(-2t)dt.$$

Although we could do this integral directly, it is instructive to set $u = t^2$ so $du = 2tdt$ in which case

$$I = \int_{0^2}^{1^2} udu + u(1-u)(-1)du = \int_0^1 u^2 du = \frac{1}{3}.$$

(b) $x'(t) = \cos t$ and $y'(t) = -\cos t$ so

$$I = \int_0^{\frac{\pi}{2}} \sin t \cos t dt + \sin t(1-\sin t)(-\cos t)dt.$$

An obvious substitution to make is $u = \sin t$ so $du = \cos t dt$ in which case

$$I = \int_{\sin 0}^{\sin\frac{\pi}{2}} udu + u(1-u)(-1)du = \int_0^1 u^2 du = \frac{1}{3}.$$

This provides an example of the fact that the value of line integrals are independent of their parameterization. It also shows how a proof of the general result could be constructed.

(5) (a) $\int_0^1 (x+z)x' + zy' + (y+x)z'dt = \int_0^1 (t+t^3)(1) + t^3(2t) + (t+t^2)(3t^2)dt$

$$\int_0^1 t + t^3 + 2t^4 + 3t^3 + 3t^4 dt$$
$$= \frac{1}{2}t^2 + \frac{1}{4}t^4 + \frac{2}{5}t^5 + \frac{3}{4}t^4 + \frac{3}{5}t^5|_0^1$$
$$= \frac{1}{2} + \frac{1}{4} + \frac{2}{5} + \frac{3}{4} + \frac{3}{5} = \frac{5}{2}$$

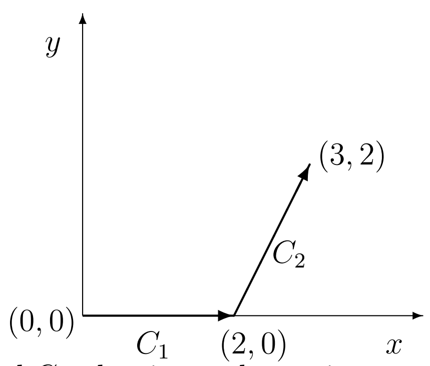
(b)

$$\frac{\partial f}{\partial x} = x + z \Rightarrow \qquad f = \frac{1}{2}x^2 + xz + c_1(y,z)$$
$$\frac{\partial f}{\partial y} = z \Rightarrow \qquad f = yz + c_2(x,z)$$
$$\frac{\partial f}{\partial z} = x + y \Rightarrow \qquad f = xz + yz + c_3(x,y)$$
$$\Rightarrow f = \frac{1}{2}x^2 + yz + xz + c$$

(c) Since $\int \mathbf{F} \cdot \mathbf{dr} = \int \nabla f \cdot \mathbf{dr} = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0))$, $\mathbf{r}(t_1) = 1\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$ and $\mathbf{r}(t_0) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ we have $\int \mathbf{F} \cdot \mathbf{dr} = \frac{1}{2} + 1 + 1 = \frac{5}{2}$

(d) Since $\mathbf{r}(t_1) = 1\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$ and $\mathbf{r}(t_0) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ we have $\int \mathbf{F} \cdot \mathbf{dr} = \frac{1}{2} + 1 + 1 = \frac{5}{2}$. Recall that the integral $\int \mathbf{F} \cdot \mathbf{dr}$ only depends on the end points if $\mathbf{F} = \nabla f$ for some f.

(6) Path $C = C_1 \cup C_2$.



First we parametrise C_1 and C_2; that is, we determine a position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.

$C_1: \quad x(t) = t, \quad y(t) = 0: \qquad 0 \le t \le 2.$
$\qquad \mathbf{r}(t) = t\mathbf{i} \Rightarrow \mathbf{r}'(t) = \mathbf{i}.$

$C_2: \quad x(t) = t, \quad y(t) = 2t - 4: \quad 2 \le t \le 3.$
$\qquad \mathbf{r}(t) = t\mathbf{i} + (2t-4)\mathbf{j} \Rightarrow \mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}.$

We use $\int_C (F_1 dx + F_2 dx) = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_{C_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt + \int_{C_2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$.
Here $\mathbf{F} = xy\mathbf{i} + (x-y)\mathbf{j}$.

Over C_1, $\mathbf{F}(\mathbf{r}(t)) = 0\mathbf{i} + (t-0)\mathbf{j} = t\mathbf{j}$ and $\mathbf{r}'(t) = \mathbf{i}$. Hence $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$. So $\int_{C_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = 0$.

Over C_2, $\mathbf{F}(\mathbf{r}(t)) = t(2t-4)\mathbf{i} + (t-(2t-4))\mathbf{j} = (2t^2-4t)\mathbf{i} + (-t+4)\mathbf{j}$ and $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}$.
Hence $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 2t^2 - 4t + 2(-t+4) = 2t^2 - 6t + 8$.

$$\int_{C_2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_2^3 (2t^2 - 6t + 8)dt = \left[\frac{2}{3}t^3 - 3t^2 + 8t\right]_2^3$$
$$= \left(\frac{2}{3} \cdot 27 - 3 \cdot 9 + 8 \cdot 3\right) - \left(\frac{2}{3} \cdot 8 - 3 \cdot 4 + 8 \cdot 2\right) = \frac{17}{3}.$$

Thus $\int_C (F_1 dx + F_2 dx) = 0 + \frac{17}{3} = \frac{17}{3}$.

Note: in this case, it is unnecessary to parametrise the curve with respect to the parameter t, because x is already a suitable parameter. However, we have preferred to illustrate the more general approach.

(7) Setting $F_1 = 3x^2y^2$ and $F_2 = 2x^3y$ we see that $\frac{\partial F_1}{\partial y} = 6x^2y = \frac{\partial F_2}{\partial x}$.

This implies $\bar{F} = (F_1, F_2)$ is a conservative vector field. This means there exist a scalar field f such that

$$\bar{F} = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$$

First $\frac{\partial f}{\partial x} = 3x^2y^2$ Integrating this we get

$$f(x,y) = x^3y^2 + g(y)$$

Differentiating this and setting equal to F_2 we get

$$\frac{\partial f}{\partial y} = 2x^3y + g'(y) = F_2 = 2x^3y$$

This means that $g'(y) = 0$. Integrating we have $g(x) = constant = k$ and
$f(x,y) = x^3y^2 + k$. The fundumental theorem of line integrals is :

$$\int_A^B \nabla f.dr = f(B) - f(A)$$

In our case $A = (0,0)$ and $B = (1,0)$ and so

$$\int_C 3x^2y^2dx + 2x^3ydy = f(1,0) - f(0,0) = 0 - 0 = 0$$