DEPARTMENT OF MATHEMATICS

MATH2000
Curl and Stokes' theorem

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(1) Calculate the curl of the vector field $\boldsymbol{F} = y^2z^3\boldsymbol{i} + x^4z^5\boldsymbol{j} + x^6y^7\boldsymbol{k}$.

(2) The velocity potential for vortex flow is $\phi = -\dfrac{\gamma}{2\pi}\tan^{-1}\left(\dfrac{y}{x}\right)$. Use your answer to Q1 from the problem sheet on "Conservative vector fields and line integrals" to show that the flow is irrotational (i.e. $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$). The field appears to be rotating, why is it called "irrotational"?

(3) If $\boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{A}$ for a vector field $\boldsymbol{A} = \boldsymbol{A}(x, y, z)$, then show $\boldsymbol{\nabla}.\boldsymbol{F} = 0$.

Note: this question is asking to prove the result $\text{div}(\text{curl}(\boldsymbol{A})) = 0$.

(4) Prove that a conservative vector field is irrotational (assume suitable differentiability for any functions involved).

Note: this question is asking to prove the result $\text{curl}(\text{grad}(f)) = \mathbf{0}$.

(5) Evaluate $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 8$ that lies inside the cylinder $x^2+y^2 = 4$ for $z > 0$ and $\mathbf{F} = xz\mathbf{i}+yz\mathbf{j}+x^2y^2\mathbf{k}$. Hint: Use Stokes' Theorem.

(6) Let $\boldsymbol{F}(x, y, z) = (xz + y)\boldsymbol{i} + (xz^3 + zy)\boldsymbol{j} + xyz\boldsymbol{k}$. Use Stokes' theorem to evaluate

$$\oint_C \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

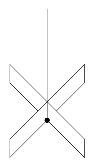
where C is the curve $x^2 + y^2 = 4$ with $z = 2$. Take C to be anticlockwise when viewed from the perspective of $z > 2$.

(7) Find the work done by a force $\boldsymbol{F} = 4z\boldsymbol{i} - 2y\boldsymbol{j} + 2y\boldsymbol{k}$ around the ellipse described by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = x + 1$. Take the direction to be anticlockwise around the ellipse as seen from an observer above the curve (say at the point $(x, y, z) = (0, 0, 2)$).

(8) The motion of a fluid is described by the velocity field $\boldsymbol{v} = xe^{-y}\boldsymbol{i} + xz\boldsymbol{j} + ze^{y}\boldsymbol{k}$.

(a) Calculate the curl of $\boldsymbol{v}$.

(b) Say you have a small paddlewheel attached to a rod, so that the wheel is allowed to rotate about the axis of the rod.



If you place the wheel into the fluid so the rod lies along the x-axis with the wheel at the point $(1, 0, 0)$, estimate whether the wheel will rotate clockwise, anticlockwise or not at all, as viewed from the perspective of the origin.

What if the rod is placed along the y-axis, so the wheel is at $(0, 1, 0)$?

What if the rod is placed along the z-axis, so the wheel is at $(0, 0, 1)$?

(9) Let $\boldsymbol{F} = \left(-\frac{1}{3}x^3 - 3xz^2\right)\boldsymbol{i} + x^2y\boldsymbol{j} + z^3\boldsymbol{k}$.

(a) Show that the divergence of $\boldsymbol{F}$ is zero.

(b) Note that $\boldsymbol{F}$ is defined everywhere in $\mathbb{R}^3$, so every closed surface in the domain of $\boldsymbol{F}$ encloses points which are also in the domain of $\boldsymbol{F}$. This fact, along with the result of part (a), tells us that there exists a vector potential $\boldsymbol{G}$ such that $\boldsymbol{F} = \text{curl}\boldsymbol{G}$.

Let f be a continuous, differentiable scalar function on $\mathbb{R}^3$. Let $\boldsymbol{G}_0 = xz^3\boldsymbol{j} - \frac{1}{3}x^3y\boldsymbol{k}$. Verify that the vector field

$$\boldsymbol{G}(x, y, z) = \boldsymbol{G}_0(x, y, z) + \nabla f(x, y, z)$$

is indeed a vector potential for $\boldsymbol{F}$, by calculating $\text{curl}\boldsymbol{G}$.

The freedom to choose the function f in this context is known as *Gauge freedom.*

(c) Use the result of part (b) to show that the flux (directed downwards towards the x-y plane) of $\boldsymbol{F}$ through the surface of the cone $z^2 = x^2 + y^2$ for $0 \le z \le 1$ is equal to $-\pi$. Hint: use Stokes' theorem, being careful to choose the correct orientation of the boundary curve.