DEPARTMENT OF MATHEMATICS

MATH2000
Curl and Stokes' theorem (solutions)

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(1)

$$\begin{aligned}
\text{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & x^4z^5 & x^6y^7 \end{vmatrix} \\
&= \boldsymbol{i}(7x^6y^6 - 5x^4z^4) - \boldsymbol{j}(6x^5y^7 - 3y^2z^2) + \boldsymbol{k}(4x^3z^5 - 2yz^3).
\end{aligned}$$

(2) The velocity obtained previously was $\boldsymbol{v} = \frac{\gamma y}{2\pi(x^2+y^2)}\boldsymbol{i} - \frac{\gamma x}{2\pi(x^2+y^2)}\boldsymbol{j}$.

To show irrotation:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{v} &= (\frac{\partial}{\partial x}(-\frac{\gamma x}{2\pi(x^2+y^2)}) - \frac{\partial}{\partial y}(\frac{\gamma y}{2\pi(x^2+y^2)}))\boldsymbol{k} \\
&= \frac{-\gamma}{2\pi}(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2})\boldsymbol{k} \\
&= \frac{-\gamma}{2\pi}(\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} + \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2})\boldsymbol{k} \\
&= 0
\end{aligned}$$

The fluid is irrotional point-wise (i.e. at each point) rather than as a whole. If you place a tiny paddle wheel in the fluid at any point it would not spin.

(3)

$$\boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

$$\Rightarrow \boldsymbol{F} = (\frac{\partial A_2}{\partial z} - \frac{\partial A_3}{\partial y})\boldsymbol{i} + (\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z})\boldsymbol{j} + (\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x})\boldsymbol{k}$$

$$\begin{aligned}
\Rightarrow \boldsymbol{\nabla}.\boldsymbol{F} &= (\frac{\partial^2 A_2}{\partial z\partial x} - \frac{\partial^2 A_3}{\partial y\partial x}) + (\frac{\partial^2 A_3}{\partial x\partial y} - \frac{\partial^2 A_1}{\partial z\partial y}) + (\frac{\partial^2 A_1}{\partial y\partial z} - \frac{\partial^2 A_2}{\partial x\partial z}) \\
&= 0
\end{aligned}$$

Since $\frac{\partial^2 g(x,y,z)}{\partial z\partial x} = \frac{\partial^2 g(x,y,z)}{\partial x\partial z}$ *etc* for well behaved functions g (prove it!)

(4) A vector field $\mathbf{v}$ is said to be irrotational if curl $\mathbf{v} = \mathbf{0}$. Suppose $\mathbf{v}$ is a conservative vector field. For some function $f(x,y,z)$,

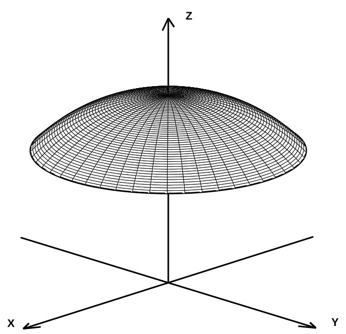
$$\mathbf{v} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Hence

$$\begin{aligned}
\operatorname{curl}\mathbf{v} &= \operatorname{curl}\left(\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\mathbf{i}+\frac{\partial \mathrm{f}}{\partial \mathrm{y}}\mathbf{j}+\frac{\partial \mathrm{f}}{\partial \mathrm{z}}\mathbf{k}\right)\\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{vmatrix}\\
&= \left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)\mathbf{i}-\left(\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial^2 f}{\partial z\partial x}\right)\mathbf{j}\\
&\quad+\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)\mathbf{k}
\end{aligned}$$

which, assuming all the second derivatives are continuous, equals $\mathbf{0}$. Hence $\mathbf{v}$ is irrotational.

(5) Sketch the region!



The boundary curve of the surface is found by taking $x^2+y^2=4$ on the sphere surface so $4+z^2=8$ or $z=\pm 2$. Since $z>0$ we have $z=2$, so the curve is the circle $x^2+y^2=4$ with $z=2$. The vector equation of the curve is

$$r(t)=2\cos t\mathbf{i}+2\sin t\mathbf{j}+2\mathbf{k}$$

$$r'(t)=-2\sin t\mathbf{i}+2\cos t\mathbf{j}+0\mathbf{k}.$$

By Stokes' Theorem $\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=\oint\mathbf{F}(\mathbf{r}(\mathbf{t}))\cdot\mathbf{r}'(t)dt$

$$=\int_0^{2\pi}(2\cos t\times 2\mathbf{i}+2\sin t\times 2\mathbf{j}+2\cos^2 t\sin^2 t\mathbf{k})\cdot(-2\sin t\mathbf{i}+2\cos t\mathbf{j}+0\mathbf{k})dt$$

$$= \int_0^{2\pi} -8\cos t \sin t + 8 \cos t \sin t dt = 0.$$

(6) By Stokes' Theorem, $\oint \mathbf{F}(\mathbf{r}(\mathbf{t})) \cdot \mathbf{r}'(t)dt = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS$. Since $z = 2$ we have

$$\begin{aligned}
\text{curl}\boldsymbol{F} &= \nabla \times \big((xz+y)\mathbf{i} + (xz^3+zy)\mathbf{j} + (xyz)\mathbf{k}.\big) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz+y & xz^3+zy & xyz \end{vmatrix} \\
&= \big(xz - 3xz^2 - y\big)\,\mathbf{i} - (yz - x)\,\mathbf{j} + \big(z^3 - 1\big)\,\mathbf{k} \\
&= (-10x - y)\,\mathbf{i} - (2y - x)\,\mathbf{j} + (7)\,\mathbf{k} \text{ when restricting to } z = 2.
\end{aligned}$$

Note that there are many surfaces with C as the boundary, it is clear that the surface defined by the flat disc with $r = 2$ is the simplest. The appropriately directed unit normal to the disc is $\mathbf{k}$ (by the right hand rule) so the RHS becomes

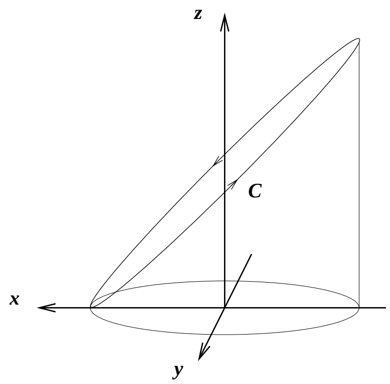
$$\iint_S \big(()\,\mathbf{i} - ()\,\mathbf{j} + (7)\,\mathbf{k}\big) \cdot \mathbf{k}\, dS = \iint_S 7dS = 7\iint_S dS = 7\pi 2^2 = 28\pi$$

since the region S is a disc of radius 2, and $\iint dS$ =Area.

(7) The work done by $\boldsymbol{F}$ around C is

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{dr} = \iint_S \text{curl}\boldsymbol{F} \cdot \boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is the unit normal vector such that the orientation of C is positive, by Stokes' theorem.



In this case,

$$\begin{aligned}
\text{curl}\boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4z & -2y & 2y. \end{vmatrix} \\
&= \boldsymbol{i}(2) - \boldsymbol{j}(-4) + \boldsymbol{k}(0) \\
&= 2\boldsymbol{i} + 4\boldsymbol{j}.
\end{aligned}$$

Take the surface S to be in the plane $z = x + 1$ bounded by C: $x^2 + y^2 = 1$. In other words,

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + (x + 1)\boldsymbol{k}$$

traces out the surface as x and y vary, provided $x^2 + y^2 \leq 1$. This surface is represented more simply by cylindrical coordinates. Setting $x = r\cos\theta$, $y = r\sin\theta$, we have the parametrisation

$$\boldsymbol{r}(r, \theta) = r\cos\theta\boldsymbol{i} + r\sin\theta\boldsymbol{j} + (r\cos\theta + 1)\boldsymbol{k}$$

over the region

$$D = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}.$$

Hence,

$$\iint_S \mathrm{curl}\boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint_D \mathrm{curl}\boldsymbol{F} \cdot (\boldsymbol{r}_r \times \boldsymbol{r}_\theta)\ dr\ d\theta$$

provided $\boldsymbol{n}$ and $\boldsymbol{r}_r \times \boldsymbol{r}_\theta$ have the same direction.

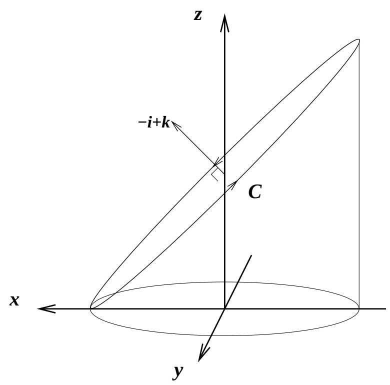
The tangent vectors are

$$\begin{aligned} \boldsymbol{r}_r &= \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} + \cos\theta\boldsymbol{k} \\ \boldsymbol{r}_\theta &= -r\sin\theta\boldsymbol{i} + r\cos\theta\boldsymbol{j} - r\sin\theta\boldsymbol{k}. \end{aligned}$$

Hence

$$\begin{aligned} \boldsymbol{r}_r \times \boldsymbol{r}_\theta &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & \cos\theta \\ -r\sin\theta & r\cos\theta & -r\sin\theta \end{vmatrix} \\ &= \boldsymbol{i}(-r\sin^2\theta - r\cos^2\theta) - \boldsymbol{j}(0) + \boldsymbol{k}(r\cos^2\theta + r\sin^2\theta) \\ &= -r\boldsymbol{i} + r\boldsymbol{k}, \end{aligned}$$

and the direction is okay (remember the "right hand rule").



We have

$$\Rightarrow\ \mathrm{curl}\boldsymbol{F} \cdot (\boldsymbol{r}_r \times \boldsymbol{r}_\theta) = (2\boldsymbol{i} + 4\boldsymbol{j}) \cdot (-r\boldsymbol{i} + r\boldsymbol{k}) = -2r.$$

The work done is then

$$
\begin{aligned}
&= \int_0^1 \int_0^{2\pi} -2r \, d\theta \, dr \text{ (Stokes' theorem)} \\
&= \left(\int_0^{2\pi} d\theta \right) \left(-\int_0^1 2r \, dr \right) \\
&= -2\pi \left[r^2 \right]_0^1 = -2\pi.
\end{aligned}
$$

(8)

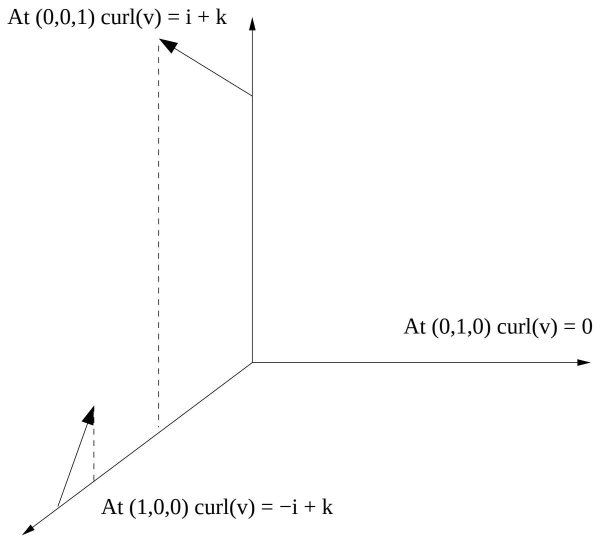
$$\vec{v} = xe^{-y}\vec{i} + xz\vec{j} + ze^{y}\vec{k}$$

(a)

$$\text{curl}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{-y} & xz & ze^{y} \end{vmatrix} = (ze^{y} - x)\vec{i} - (0 - 0)\vec{j} + (z + xe^{y})\vec{k}$$

$$\text{curl}\vec{v} = (ze^{y} - x)\vec{i} + (z + xe^{y})\vec{k}$$

(b) At $(1,0,0)$, $\text{curl}\vec{v} = -\vec{i} + \vec{k}$ and $(\text{curl}\vec{v}) \cdot \vec{i} = -1$. Viewing from the origin, since the $\vec{i}$ component of $\text{curl}\vec{v}$ is negative, the wheel rotates anticlockwise. At $(0,1,0)$, $\text{curl}\vec{v} = 0$ and $(\text{curl}\vec{v}) \cdot \vec{j} = 0$, so the wheel does not rotate. At $(0,0,1)$, $\text{curl}\vec{v} = \vec{i} + \vec{k}$ and $(\text{curl}\vec{v}) \cdot \vec{k} = 1$. Viewing from the origin, since the $\vec{k}$ component of $\text{curl}\vec{v}$ is positive, the wheel rotates clockwise.



(9) (a)

$$\vec{F} = (-\frac{1}{3}x^3 - 3xz^2)\vec{i} + x^2y\vec{j} + z^3\vec{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(-\frac{1}{3}x^3 - 3xz^2\right) + \frac{\partial}{\partial y}\left(x^2y\right) + \frac{\partial}{\partial z}\left(z^3\right)$$

$$= -x^2 - 3z^2 + x^2 + 3z^2 = 0$$

(b)

$$\vec{G} = \vec{G}_0 + \nabla f(x,y,z) = \vec{i}\left(\frac{\partial f}{\partial x}\right) + \vec{j}\left(xz^3 + \frac{\partial f}{\partial y}\right) + \vec{k}\left(-\frac{1}{3}x^3y + \frac{\partial f}{\partial z}\right)$$

$$\text{curl}\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & xz^3 + \frac{\partial f}{\partial y} & -\frac{1}{3}x^3y + \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(-\frac{1}{3}x^3 - 3xz^2 + \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial x}\right) - \vec{j}\left(-x^2y + \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)$$

$$+\vec{k}\left(z^3 + \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$$= (-\frac{1}{3}x^3 - 3xz^2)\vec{i} + x^2y\vec{j} + z^3\vec{k} = F$$

(c)

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_S (\text{curl}\vec{G}) \cdot \vec{n} dS = \oint \vec{G} \cdot d\vec{r}$$

$$\oint_C \vec{G} \cdot d\vec{r} = \oint_C \vec{G}_0 \cdot d\vec{r} + \oint_C \nabla f \cdot d\vec{r}$$

From the fundumental theorem of line integrals

$$\oint_C \nabla f \cdot d\vec{r} = 0 \text{ giving } \oint_C \vec{G} \cdot d\vec{r} = \oint_C \vec{G}_0 \cdot d\vec{r}$$

The curve C is a circle in the plane z=1, radius 1, centre on the z axis. Choosing a parameterization $\vec{r}(t) = \cos(t)\vec{i} - \sin(t)\vec{j} + \vec{k}$ with $0 \leq t \leq 2\pi$ (note the direction of C!).

$$\vec{G}_0(\vec{r}(t)) = \cos(t)\vec{j} + \frac{1}{3}\cos^3(t)\sin(t)\vec{k}$$

$$\frac{d\vec{r}(t)}{dt} = -\sin(t)\vec{i} - \cos(t)\vec{j}$$

$$\vec{G}_0(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt} = -\cos^2(t) = -\frac{1}{2} - \frac{1}{2}\cos(2t)$$

$$\oint_C \vec{G}_0 \cdot d\vec{r} = -\int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(2t)dt = -\left[\frac{1}{2}t\right]_0^{2\pi} = -\pi$$

NOTE :

Evaluating the flux integral $\iint_S \vec{F} \cdot \vec{n}dS$ could be difficult. You would need a parameterisation

$$\vec{r}(r, \theta) = \frac{r\cos(\theta)}{\sqrt{2}}\vec{i} + \frac{r\sin(\theta)}{\sqrt{2}}\vec{j} + \frac{r}{\sqrt{2}}\vec{k}$$

with $0 \leq r \leq \sqrt{2}$, $0 \leq \theta \leq 2\pi$ for the surface of the cone, then use

$$\vec{r}_\theta \times \vec{r}_r = \frac{1}{2}r\cos(\theta)\vec{i} + \frac{1}{2}r\sin(\theta)\vec{j} - \frac{1}{2}r\vec{k}.$$

Then we have

$$\vec{F}(\vec{r}(r,\theta)) \cdot (\vec{r}_\theta \times \vec{r}_r) = -\alpha^5 r^4 \cos^2(\theta)\left(\frac{\cos^2(\theta)}{3} + 3\right) + \alpha^5 r^4 \cos^2(\theta)\sin^2(\theta) - \alpha^5 r^4$$

where $\alpha = \frac{1}{2^{\frac{3}{2}}}$. You would then verify that

$$\int_0^{\sqrt{2}} \int_0^{2\pi} \left(-\alpha^5 r^4 \cos^2(\theta)\left(\frac{\cos^2(\theta)}{3} + 3\right) + \alpha^5 r^4 \cos^2(\theta)\sin^2(\theta) - \alpha^5 r^4\right)\, d\theta\, dr = -\pi.$$