DEPARTMENT OF MATHEMATICS

MATH2000
Double Integrals in Rectangular Coordinates solutions.

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(1) (a)

$$\begin{aligned}
\int_0^2 \int_0^1 (x+y)\ dx\ dy &= \int_0^2 \left[\frac{1}{2}x^2 + xy\right]_{x=0}^{x=1} dy \\
&= \int_0^2 (\frac{1}{2} + y)\ dy \\
&= \left[\frac{1}{2}y + \frac{1}{2}y^2\right]_{y=0}^{y=2} \\
&= \frac{2}{2} + \frac{4}{2} = 3.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^1 \int_0^2 (x^4y^5 + y)dx\ dy &= \int_0^2 \left[\frac{1}{5}x^5y^5 + xy\right]_{x=0}^{x=2} dy \\
&= \int_0^2 \left(\frac{32}{5}y^5 + 2y\right) dy \\
&= \left[\frac{32y^6}{30} + y^2\right]_{y=0}^{y=1} \\
&= \frac{16}{15} + 1 = \frac{31}{15}.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^1 \int_0^{2x} 2y^2\ dy\ dx &= \int_0^1 \left[\frac{2y^3}{3}\right]_{y=0}^{y=2x} dx \\
&= \int_0^1 \frac{16x^3}{3} dx \\
&= \left[\frac{4x^4}{3}\right]_{x=0}^{x=1} \\
&= \frac{4}{3}
\end{aligned}$$

(d)

$$\begin{aligned}
\int_0^1 \int_y^{\sqrt{y}} 2x \; dx \; dy &= \int_0^1 \left[x^2\right]_{x=y}^{x=\sqrt{y}} dy \\
&= \int_0^1 (y - y^2) dy \\
&= \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_{x=0}^{x=1} \\
&= \frac{1}{6}
\end{aligned}$$

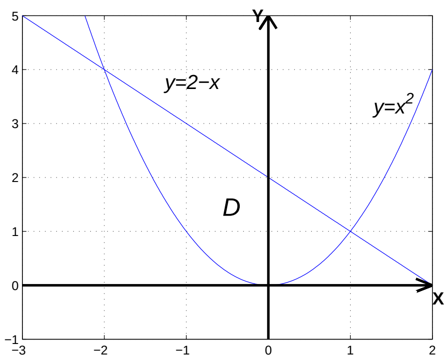
(2) The region in the x-y plane can be expressed as a type I region:

$$D = \{(x, y) | 0 \le x \le 4, 0 \le y \le x\}$$

$$\begin{aligned}
\Rightarrow \text{Volume} &= \int_0^4 \int_0^x (x - y) \; dy \; dx \\
&= \int_0^4 \left[xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx \\
&= \int_0^4 \frac{x^2}{2} dx \\
&= \left[\frac{x^3}{6}\right]_{x=0}^{x=4} \\
&= \frac{32}{3}.
\end{aligned}$$

Note: It's a pyramid. Volume $= \frac{1}{3}bh$. The base is a triangle of area 8, the height is 4, which gives the same answer.

(3) To start with, it is useful to draw a diagram. The following was done in MATLAB:



It is straightforward to determine that the two curves intersect when

$$x^2 = 2 - x \quad \Rightarrow \quad x = -2, 1.$$

The region in the x-y plane bounded by these two curves is represented more easily as a type I region (ie. bounded by two constant x values):

$$D = \left\{(x, y) \mid -2 \le x \le 1,\ x^2 \le y \le 2 - x\right\}.$$

From this we can read off the bounds of integration and determine the double integral as the following iterated integral:

$$\begin{aligned}
\iint_D x\, dA &= \int_{-2}^{1} \int_{x^2}^{2-x} x\, dy\, dx \\
&= \int_{-2}^{1} [xy]_{x^2}^{2-x}\, dx \\
&= \int_{-2}^{1} \left(x(2-x) - x \cdot x^2\right) dx \\
&= \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{-2}^{1} \\
&= \left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left((-2)^2 - \frac{1}{3}(-2)^3 - \frac{1}{4}(-2)^4\right) \\
&= \frac{12 - 4 - 3}{12} - \left(\frac{48 + 32 - 48}{12}\right) = -\frac{27}{12} = -\frac{9}{4}.
\end{aligned}$$

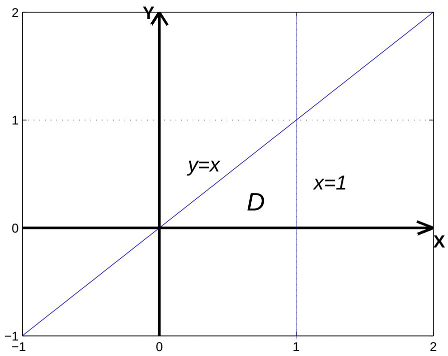
Note: a common mistake is to get a sign wrong in expanding the second last line above.

Also, a common problem students have is interpreting the negative answer. Remember, the double integral really gives the "net volume" above the x-y plane. In this case, since the function we are integrating ($f(x, y) = x$) defines a surface which lies below the x-y plane for negative values of x, we can interpret the negative answer to mean that there is more volume of the solid below the x-y plane than there is above.

(4) Since we cannot integrate $\cos(x^2)$ with respect to x, we can try changing the order of integration. The region of integration in the x-y plane is given by

$$D = \{(x, y) \mid 0 \le y \le 1,\ y \le x \le 1\}.$$

To clearly see how to change the order of integration, it may be useful to draw a diagram. The following was done in MATLAB:



From the diagram above, it should be clear that we can also represent the region of integration as

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}.$$

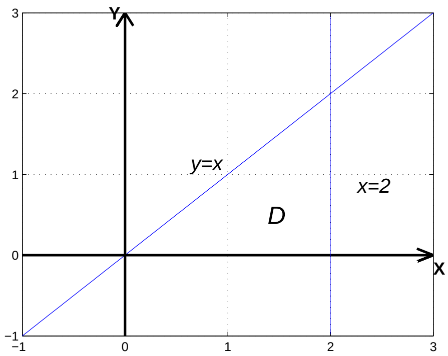
We then have

$$\begin{aligned}
\int_0^1 \int_y^1 \cos(x^2)\ dx\ dy &= \int_0^1 \int_0^x \cos(x^2)\ dy\ dx \\
&= \int_0^1 \left[y\cos(x^2)\right]_0^x\ dx \\
&= \int_0^1 x\cos(x^2)\ dx \\
&= \left[\frac{1}{2}\sin(x^2)\right]_0^1 = \frac{1}{2}\sin(1).
\end{aligned}$$

(5)

$$z = f(x, y) = e^{-x^2}$$

Region of integration:



This is either a type I or II region, but since e^{-x^2} isn't integrable with respect to x, choose type I.

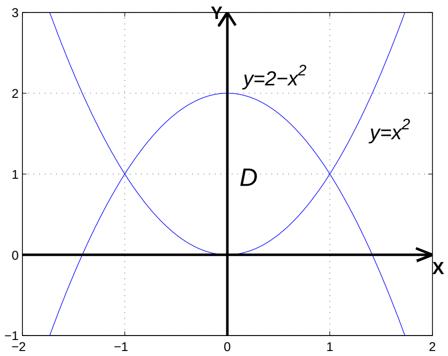
$$D = \{(x,y)|0 \le x \le 2, 0 \le y \le x\}$$

$$\begin{aligned}\Rightarrow \text{Vol.} &= \int_0^2 \int_0^x e^{-x^2} dy dx \\ &= \int_0^2 x e^{-x^2} dx\end{aligned}$$

Set $u = x^2 \Rightarrow du = 2x\, dx$

$$\begin{aligned}\text{Vol.} &= \int_0^4 \frac{1}{2} e^{-u} du \\ &= -\frac{1}{2}\left[e^{-u}\right]_0^4 \\ &= \frac{1}{2}\left(1 - \frac{1}{e^4}\right)\end{aligned}$$

(6) The region:



Choosing type I:

$$D = \{(x,y)| -1 \le x \le 1, x^2 \le y \le 2 - x^2\}$$

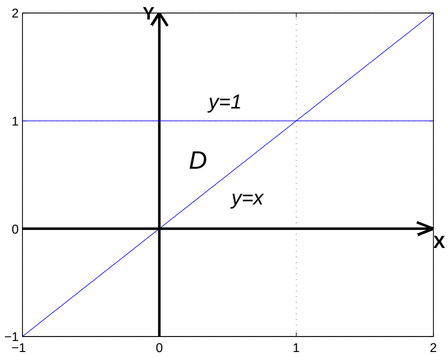
$$
\begin{aligned}
\text{Vol.} &= \int_{-1}^{1}\int_{x^2}^{2-x^2} (x+y)dydx \\
&= \int_{-1}^{1} \left[xy + \frac{y^2}{2}\right]_{x^2}^{2-x^2} dx \\
&= \int_{-1}^{1} \left((2x - x^3 + \frac{1}{2}(4 - 4x^2 + x^4)) - (x^3 + \frac{x^4}{2})\right) dx \\
&= \int_{-1}^{1} 2 + 2x - 2x^2 - 2x^3 \, dx \\
&= \left[2x + x^2 - \frac{2x^3}{3} - \frac{x^4}{2}\right]_{-1}^{1} \\
&= (2 + 1 - \frac{2}{3} - \frac{1}{2}) - (-2 + 1 + \frac{2}{3} - \frac{1}{2}) \\
&= \frac{8}{3}
\end{aligned}
$$

Strictly speaking, this is the *net* volume, since $z < 0$ for $x + y < 0$.

(7)

$$\int_0^1 \int_x^1 f(x,y) \; dy \; dx$$

$$\Rightarrow D = \{(x,y)|0 \leq x \leq 1, x \leq y \leq 1\}$$



This can be rewritten as:

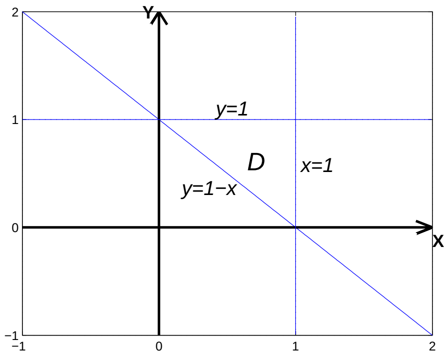
$$D = \{(x,y)|0 \leq y \leq 1, 0 \leq x \leq y\}$$

$$\Rightarrow \text{ ans. is } \int_0^1 \int_0^y f(x,y) \; dx \; dy$$

(8)

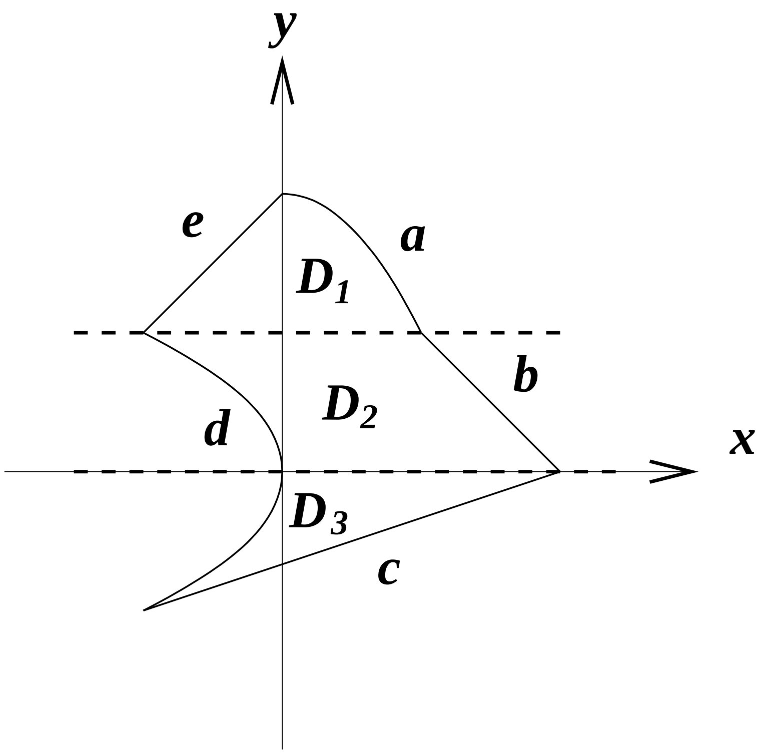
$$\int_0^1 \int_{1-x}^1 f(x,y)\ dy\ dx$$

$$\Rightarrow D = \{(x,y)|0 \le x \le 1, 1-x \le y \le 1\}$$



As type II: $D = \{(x,y)|0 \le y \le 1, 1-y \le x\ \ \le 1\} \Rightarrow$ ans. is $\displaystyle\int_0^1 \int_{1-y}^1 f(x,y)\ dx\ dy$

(9)



$$D_1 = \{(x,y)|1 \le y \le 2, y-2 \le x \le \sqrt{2-y}\}$$
$$D_2 = \{(x,y)|0 \le y \le 1, -y^2 \le x \le 2-y\}$$
$$D_3 = \{(x,y)|-1 \le y \le 0, -y^2 \le x \le 3y+2\}$$

$$\begin{aligned}\iint_D f(x,y)dA &= \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA + \iint_{D_3} f(x,y)dA \\ &= \int_1^2 \int_{y-2}^{\sqrt{2-y}} f(x,y)dxdy + \int_0^1 \int_{-y^2}^{2-y} f(x,y)dxdy + \int_{-1}^0 \int_{-y^2}^{3y+2} f(x,y)dxdy\end{aligned}$$