DEPARTMENT OF MATHEMATICS

MATH2000
Divergence, parametrising surfaces and surface integrals

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(1) Calculate the divergence of the Newtonian gravitational field

$$\mathbf{F}(x,y,z) = -mMG\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}\right).$$

(2) Let $\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = ||\boldsymbol{r}|| = \sqrt{x^2+y^2+z^2}$

(a) Show that $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$.

(b) Show that $\mathrm{div}(r^n\mathbf{r}) = (3+n)r^n$.

(3) Show that

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3} \qquad \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x^2}{r^5}.$$

From your answer argue that

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3} \qquad \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3}+\frac{3y^2}{r^5} \qquad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3} \qquad \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3}+\frac{3z^2}{r^5}$$

Hence show that $\nabla\cdot(\nabla\frac{1}{r}) \equiv \nabla^2\frac{1}{r} = 0$.

(4) Use the flux form of Green's theorem to calculate the net outward flux of $\boldsymbol{v} = (x+y)\boldsymbol{i} - (x^2+y^2)\boldsymbol{j}$ across the boundary of the triangle with vertices $(1,0)$, $(0,1)$ and $(-1,0)$. Note this is the same problem as Q5 from the tutorial sheet on "Green's theorem, introduction to flux". Which method do you think is easier?

(5) Identify the surface with the given parametrisation.

(a) $\boldsymbol{r}(u,v) = (1+2u)\boldsymbol{i} + (-u+3v)\boldsymbol{j} + (2+4u+5v)\boldsymbol{k}$,

(b) $\boldsymbol{r}(x,\theta) = < x, x\cos\theta, x\sin\theta >$.

(6) (Stewart, Q33 p1116)

Find the equation of the tangent plane to the surface

$$x = u+v,\ y = 3u^2,\ z = u-v,$$

at the point $(2,3,0)$.

(7) (Stewart, Q41 p1116)

Find the surface area of the part of the surface $z = xy$ that lies within the cylinder $x^2+y^2 = 1$.

(8) What is the average value of the function $f(x,y,z) = x+y+z$ over the surface of the box $\{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$? Does your answer make sense?

(9) Consider the vector field $\boldsymbol{v}(x,y) = \left(xy^2 - xy\right)\boldsymbol{i} - \left(\frac{1}{2}x^2y^2 - x^2y\right)\boldsymbol{j}$.

(a) Calculate the divergence of $\boldsymbol{v}(x,y)$.

(b) Find all curves in the x-y plane over which the divergence is zero. Plot these curves and show how they partition the plane into regions of positive and negative divergence (ie. indicate which regions have positive divergence and which have negative divergence).

(c) Repeat parts (a) and (b) for the vector field

$$\boldsymbol{v}(x,y) = y\boldsymbol{i} + (x+y^3)\boldsymbol{j}.$$

(10) The surface S which is the part of the cylinder $x^2+z^2=a^2$ that lies inside the cylinder $x^2+y^2=a^2$ can be parametrised by $\boldsymbol{r}(\phi,\theta) = a\cos\theta\boldsymbol{i} + a\sin\phi\sin\theta\boldsymbol{j} + a\sin\theta\boldsymbol{k}$, $0 \le \theta \le 2\pi$, $-\pi/2 \le \phi \le \pi/2$. Find the surface area of S.

Hint: Since $0 \le \theta \le 2\pi$, take

$$\sqrt{\sin^2\theta} = \begin{cases} \sin\theta, & 0 \le \theta \le \pi; \\ -\sin\theta, & \pi \le \theta \le 2\pi, \end{cases}$$

since $\sin\theta \le 0$ for $\pi \le \theta \le 2\pi$.