DEPARTMENT OF MATHEMATICS

MATH2000
Divergence, parametrising surfaces and surface integrals (solutions).

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(1) $\mathbf{F}(x,y,z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ where

$$\begin{aligned} F_1 &= -mMG\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right), \\ F_2 &= -mMG\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right), \\ F_3 &= -mMG\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right). \end{aligned}$$

$\operatorname{div}\mathbf{F} = \frac{\partial \mathrm{F_1}}{\partial \mathrm{x}} + \frac{\partial \mathrm{F_2}}{\partial \mathrm{y}} + \frac{\partial \mathrm{F_2}}{\partial \mathrm{z}}$.

Using the quotient rule,

$$\begin{aligned} & \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) \\ = & \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\ = & \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}} \\ = & \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

and

$$\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$$

Hence $\operatorname{div}\mathbf{F} = -mMG\left(\frac{-2x^2+y^2+z^2+x^2-2y^2+z^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\right) = 0$.

(2) (a)

$$\frac{\partial r}{\partial x} = \frac{1}{2}(2x)(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{x}{r}$$

which can be even more elegantly derived implicitly by

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2) \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

(b)

$$\begin{aligned}
\operatorname{div}(r^n\mathbf{r}) &= \frac{\partial(xr^n)}{\partial x} + \frac{\partial(yr^n)}{\partial y} + \frac{\partial(zr^n)}{\partial z} \\
&= x\frac{\partial r^n}{\partial x} + r^n\frac{\partial x}{\partial x} + y\frac{\partial r^n}{\partial y} + r^n\frac{\partial y}{\partial y} + z\frac{\partial r^n}{\partial z} + r^n\frac{\partial z}{\partial z} \\
&= xnr^{n-1}\frac{x}{r} + r^n + ynr^{n-1}\frac{y}{r} + r^n + znr^{n-1}\frac{z}{r} + r^n \\
&= n(x^2+y^2+z^2)r^{n-2} + 3r^n = (n+3)r^n
\end{aligned}$$

so $\operatorname{div}(r^n\mathbf{r}) = (3+n)r^n$.

(3)

$$\begin{aligned}
\frac{\partial}{\partial x}(\frac{1}{r}) &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\frac{\partial}{\partial x}(x^2+y^2+z^2) \\
&= -\frac{1}{2}(2x)(x^2+y^2+z^2)^{-\frac{3}{2}} = -x(\sqrt{x^2+y^2+z^2})^{-3} = xr^{-3} = -\frac{x}{r^3}
\end{aligned}$$

This is rather tedious. The calcuation is simplified if you retain $r(x,y,z)$ in the calculation. Note

$$\frac{\partial r}{\partial x} = \frac{1}{2}(2x)(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{x}{r}$$

which can be even more elegantly derived implicitly by

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2) \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Now, returning to the original problem we use the chain rule to show

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{x}{r} = -\frac{x}{r^3}.$$

Now to find the second derivative we use the same method

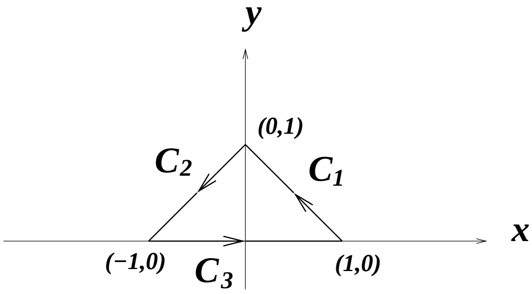
$$\frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) = -\frac{1}{r^3} - x\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\frac{\partial r}{\partial x} = -\frac{1}{r^3} + 3\frac{x}{r^4}\frac{\partial r}{\partial x} = -\frac{1}{r^3} + 3\frac{x}{r^4}\frac{x}{r} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$$

The other results follow by symmetry. Now

$$\begin{aligned}
\nabla^2\left(\frac{1}{r}\right) &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) \\
&= -\frac{1}{r^3} + \frac{3x^2}{r^5} - \frac{1}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3} + \frac{3z^2}{r^5} \\
&= -\frac{3}{r^3} + \frac{3(x^2+y^2+z^2)}{r^5} = -\frac{3}{r^3} + \frac{3}{r^3} = 0.
\end{aligned}$$

Note since $\frac{1}{r}$ and its derivatives are not defined at the origin we say $\nabla^2\left(\frac{1}{r}\right) = 0$ for $r \neq 0$. (This caveat is extremely important in more advanced courses.)

(4) The diagram below shows the path of integration: we shall choose to traverse the boundary of the rectangle in a counterclockwise direction in order to apply the flux form of Green's theorem.



The form of Green's theorem we will use is

$$\oint_C \boldsymbol{v} \cdot \boldsymbol{n}\, dS = \iint_D \nabla \cdot \boldsymbol{v}\, dx\, dy.$$

The region bounded by the triangular path can be represented using rectangular coordinates as $D = \{(x, y) \mid 0 \leq y \leq 1,\; y - 1 \leq x \leq 1 - y\}$.

The divergence of the vector field is $\nabla \cdot \boldsymbol{v} = 1 - 2y$.

Therefore the net outward flux is

$$\begin{aligned}
\iint_D \nabla \cdot \boldsymbol{v}\, dx\, dy &= \int_0^1 \int_{y-1}^{1-y} (1 - 2y)\, dx\, dy \\
&= \int_0^1 [x - 2xy]_{y-1}^{1-y}\, dy \\
&= \int_0^1 (1 - 3y + 2y^2 - (-1 + 3y - 2y^2))dy \\
&= \int_0^1 (2 - 6y + 4y^2)dy \\
&= \left[2y - 3y^2 + \frac{4}{3}y^3\right]_0^1 = 2 - 3 + \frac{4}{3} = \frac{1}{3}.
\end{aligned}$$

This method seems to be a lot less effort than calculating all three line integrals.

(5) (a) In this case the function $z(u,v)$ can be written as a linear combination of $x(u,v)$ and $y(u,v)$. Indeed,

$$\begin{aligned}
2+4u+5v &= \frac{5}{3}(-u+3v)+\frac{17}{6}(1+2u)-\frac{5}{6} \\
\Rightarrow \ z &= \frac{5}{3}y+\frac{17}{6}x-\frac{5}{6} \\
\Rightarrow \ 17x+10y-6z &= 5,
\end{aligned}$$

which is the equation of a plane.

(b) In this case we are able to express the function $x(u,v)$ in terms of $y(u,v)$ and $z(u,v)$. In this case,

$$\begin{aligned}
y^2+z^2 &= x^2\cos^2\theta + x^2\sin^2\theta \\
&= x^2(\cos^2\theta+\sin^2\theta) \\
&= x^2.
\end{aligned}$$

The equation $x^2=y^2+z^2$ describes a double cone with apex at the origin and with axis of symmetry the x-axis.

(6) To find the equation to the tangent plane at the given point, we first find a normal vector to the surface at that point (which will also be normal to the tangent plane), then give an arbitrary vector lying in the plane whose dot product with the normal vector must be zero, leading to the equation of the plane.

In this case, we have parametrisation of the surface given by the position vector

$$\boldsymbol{r}(u,v)=(u+v)\boldsymbol{i}+3u^2\boldsymbol{j}+(u-v)\boldsymbol{k}.$$

Two vectors tangent to the surface are $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ which are given by

$$\boldsymbol{r}_u=\boldsymbol{i}+6u\boldsymbol{j}+\boldsymbol{k},\ \ \boldsymbol{r}_v=\boldsymbol{i}-\boldsymbol{k}.$$

Therefore a vector normal to the surface is given by

$$\begin{aligned}
\boldsymbol{r}_u\times\boldsymbol{r}_v &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 6u & 1 \\ 1 & 0 & -1 \end{vmatrix} \\
&= \boldsymbol{i}(-6u-0)-\boldsymbol{j}(-1-1)+\boldsymbol{k}(-6u-0) \\
&= -6u\boldsymbol{i}+2\boldsymbol{j}-6u\boldsymbol{k}.
\end{aligned}$$

At the point $(2,3,0)$, we have $z=0=u-v$, so $u=v$. We also have $x=2=u+v=2u$ (since $u=v$). Therefore we must have $u=1=v$, which is also consistent with $y=3=3u^2$. Therefore the point $(2,3,0)$ corresponds to parameter values $u=v=1$.

At these parameter values, the normal vector is

$$\boldsymbol{r}_u\times\boldsymbol{r}_v=-6\boldsymbol{i}+2\boldsymbol{j}-6\boldsymbol{k}.$$

Let (x, y, z) be an arbitrary point in the tangent plane, so that the vector

$$\overrightarrow{PX} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) - (2\boldsymbol{i} + 3\boldsymbol{j}) = (x-2)\boldsymbol{i} + (y-3)\boldsymbol{j} + z\boldsymbol{k}$$

lies in the tangent plane. The equation for the tangent plane is then given by

$$\begin{aligned} (\boldsymbol{r}_u \times \boldsymbol{r}_v)\cdot \overrightarrow{PX} &= 0 \\ \Rightarrow \ -6(x-2) + 2(y-3) - 6z &= 0 \\ \Rightarrow \ 3x - y + 3z &= 3. \end{aligned}$$

(7) Use a parametrisation based on polar coordinates:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= r^2\cos\theta\sin\theta, \end{aligned}$$

for $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

$$\Rightarrow \ \boldsymbol{r}(r,\theta) = r\cos\theta\boldsymbol{i} + r\sin\theta\boldsymbol{j} + r^2\cos\theta\sin\theta\boldsymbol{k}.$$

We know that the surface area can be calculated using the surface integral:

$$\text{surface area } = \iint_S dS.$$

We can evaluate this surface integral in the usual way by working out the tangent vectors $\boldsymbol{r}_r$ and $\boldsymbol{r}_\theta$, then calculating

$$\iint_S dS = \int_0^1 \int_0^{2\pi} |\boldsymbol{r}_r \times \boldsymbol{r}_\theta| \, d\theta \, dr.$$

To this end,

$$\begin{aligned} \boldsymbol{r}_r &= \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} + 2r\cos\theta\sin\theta, \\ \boldsymbol{r}_\theta &= -r\sin\theta\boldsymbol{i} + r\cos\theta\boldsymbol{j} + r^2(\cos^2\theta - \sin^2\theta)\boldsymbol{k} \end{aligned}$$

$$\begin{aligned} \Rightarrow \ \boldsymbol{r}_r \times \boldsymbol{r}_\theta &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & 2r\cos\theta\sin\theta \\ -r\sin\theta & r\cos\theta & r^2(\cos^2\theta - \sin^2\theta) \end{vmatrix} \\ &= -r^2\sin\theta\boldsymbol{i} - r^2\cos\theta\boldsymbol{j} + r\boldsymbol{k} \\ \Rightarrow \ |\boldsymbol{r}_r \times \boldsymbol{r}_\theta| &= \sqrt{r^4\sin^2\theta + r^4\cos^2\theta + r^2} \\ &= r\sqrt{r^2+1}. \end{aligned}$$

$$\begin{aligned} \text{surface area} &= \int_0^1 \int_0^{2\pi} r\sqrt{r^2+1} d\theta \, dr \\ &= 2\pi \times \frac{1}{2}\int_1^2 u^{1/2} du \text{ subst. } u = r^2+1 \\ &= \pi \times \left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{2\pi}{3}(\sqrt{8}-1). \end{aligned}$$

(8) As seen in lectures, the average value of a function $f(x,y,z)$ over a surface S is given by $\dfrac{\iint_S f(x,y,z)\ dS}{\iint_S dS}$. The quantity in the denominator is just the surface area of S. In this case, we need the surface area of the box $\{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$, which is the sum of the area of each face, which you should be able to work out to be 22.

To work out the surface integral $\oiint\limits_S (x+y+z)dS$, we calculate the surface integral over each face.

Let face 1 (S_1) lie in the plane $x = 0$. It is the rectangle defined by $\{(y,z) \mid 0 \le y \le 2, 0 \le z \le 3\}$. The function f restricted to this surface is $f(0,y,z) = y+z$, so that the surface integral over this face is just

$$\begin{aligned}
\iint\limits_{S_1} (y+z)\ dS &= \int_0^2 \int_0^3 (y+z)\ dz\ dy \\
&= \int_0^2 \left[yz + \frac{1}{2}z^2\right]_0^3 dy \\
&= \int_0^2 \left(3y + \frac{9}{2}\right)\ dy \\
&= \left[\frac{3}{2}y^2 + \frac{9}{2}y\right]_0^2 = 6 + 9 = 15.
\end{aligned}$$

Let face 2 (S_2) lie in the plane $x = 1$. It is the rectangle defined by $\{(y,z) \mid 0 \le y \le 2, 0 \le z \le 3\}$. The function f restricted to this surface is $f(1,y,z) = 1+y+z$, so that the surface integral over this face is just

$$\begin{aligned}
\iint\limits_{S_2} (1+y+z)\ dS &= \int_0^2 \int_0^3 (1+y+z)\ dz\ dy \\
&= \int_0^2 \left[z + yz + \frac{1}{2}z^2\right]_0^3 dy \\
&= \int_0^2 \left(3 + 3y + \frac{9}{2}\right)\ dy \\
&= \left[3y + \frac{3}{2}y^2 + \frac{9}{2}y\right]_0^2 = 6 + 6 + 9 = 21.
\end{aligned}$$

Let face 3 (S_3) lie in the plane $y = 0$. It is the rectangle defined by $\{(x,z) \mid 0 \le x \le 1, 0 \le z \le 3\}$. The function f restricted to this surface is $f(x,0,z) = x+z$, so that the surface integral over this face is just

$$\begin{aligned}\iint_{S_3} (x+z)\ dS &= \int_0^1 \int_0^3 (x+z)\ dz\ dx \\ &= \int_0^1 \left[xz + \frac{1}{2}z^2\right]_0^3 dx \\ &= \int_0^1 \left(3x + \frac{9}{2}\right)\ dx \\ &= \left[\frac{3}{2}x^2 + \frac{9}{2}x\right]_0^1 = \frac{3}{2} + \frac{9}{2} = 6.\end{aligned}$$

Let face 4 (S_4) lie in the plane $y = 2$. It is the rectangle defined by $\{(x,z) \mid 0 \le x \le 1, 0 \le z \le 3\}$. The function f restricted to this surface is $f(x,2,z) = x + 2 + z$, so that the surface integral over this face is just

$$\begin{aligned}\iint_{S_4} (x+2+z)\ dS &= \int_0^1 \int_0^3 (x+2+z)\ dz\ dx \\ &= \int_0^1 \left[xz + 2z + \frac{1}{2}z^2\right]_0^3 dx \\ &= \int_0^1 \left(3x + 6 + \frac{9}{2}\right)\ dx \\ &= \left[\frac{3}{2}x^2 + 6x + \frac{9}{2}x\right]_0^1 = \frac{3}{2} + 6 + \frac{9}{2} = 12.\end{aligned}$$

Let face 5 (S_5) lie in the plane $z = 0$. It is the rectangle defined by $\{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2\}$. The function f restricted to this surface is $f(x,y,0) = x + y$, so that the surface integral over this face is just

$$\begin{aligned}\iint_{S_5} (x+y)\ dS &= \int_0^1 \int_0^2 (x+y)\ dy\ dx \\ &= \int_0^1 \left[xy + \frac{1}{2}y^2\right]_0^2 dx \\ &= \int_0^1 (2x+2)\ dx \\ &= \left[x^2 + 2x\right]_0^1 = 1 + 2 = 3.\end{aligned}$$

Let face 6 (S_6) lie in the plane $z = 3$. It is the rectangle defined by $\{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2\}$. The function f restricted to this surface is $f(x,y,3) = x + y + 3$, so that the surface integral over this face is just

$$\begin{aligned}
\iint_{S_6} (x+y+3)\ dS &= \int_0^1 \int_0^2 (x+y+3)\ dy\ dx \\
&= \int_0^1 \left[xy + \frac{1}{2}y^2 + 3y \right]_0^2 dx \\
&= \int_0^1 (2x+8)\ \ dx \\
&= \left[x^2 + 8x\right]_0^1 = 1 + 8 = 9.
\end{aligned}$$

The surface integral over the entire surface of the box is just

$$\oiint_S (x+y+z)dS = \sum_{i=1}^{6} \left(\iint_{S_i} (x+y+z)dS \right) = 15 + 21 + 6 + 12 + 3 + 9 = 66.$$

Therefore the average value of the function is $\dfrac{66}{22} = 3$.

Note that these values lie on the closed curve given by the intersection of the plane $f(x,y,z) = x+y+z = 3$ and the surface of the box. The answer is not too surprising since on the surface of the box, the maximum value is 6 at the point $(1,2,3)$ and the minimum value is 0 at $(0,0,0)$. Setting different values of the function $f(x,y,z) = x+y+z$ defines planes $x+y+z = c$ for constant $c \in [0,6]$. As we vary c from 0 to 6, the “half-way point” would be 3 which turns out to be the average value.

(9) (a)

$$\mathbf{v} = (xy^2 - xy)\mathbf{i} - (\frac{1}{2}x^2y^2 - x^2y)\mathbf{j}$$

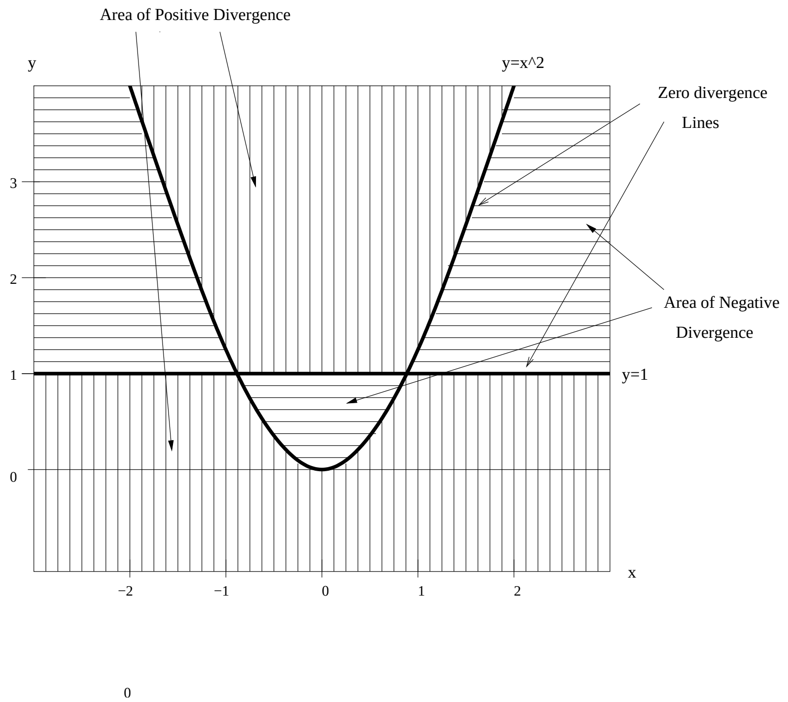
$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(xy^2 - xy) + \frac{\partial}{\partial y}(\frac{1}{2}x^2y^2 - x^2y)$$

$$= y^2 - y - x^2y + x^2 = y(y-1) - x^2(y-1)$$

$$= (y - x^2)(y - 1)$$

(b)

From part (a) $\nabla \cdot \mathbf{v} = 0$ when $y = x^2$ or $y = 1$.



(c)

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x + y^3) = 3y^2$$

$\nabla \cdot \mathbf{v} = 0$ when $y = 0$. Also $\nabla \cdot \mathbf{v} > 0$ for all other points (x, y) not on $y = 0$.

<u>Comment :</u> If a closed curve straddles regions of positive and negative divergence, it is possible that the net outward flux across the closed curve is zero (only in this case is it possible). That can never happen for the field in part (c).

(10)

The surface S is given parametrically by

$$\mathbf{r}(\phi,\theta) = a\cos(\theta)\mathbf{i} + a\sin(\phi)\sin(\theta)\mathbf{j} + a\sin(\theta)\mathbf{k}, -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi$$

$$\text{surface area} = \iint_S dS = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| d\phi d\theta$$

$$\mathbf{r}_\phi = a\cos(\phi) sin(\theta)\mathbf{j}$$

$$\mathbf{r}_\theta = -a\sin(\theta)\mathbf{i} + a\sin(\phi) cos(\theta)\mathbf{j} + a\cos(\theta)\mathbf{k}$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & a\cos(\phi)\sin(\theta) & 0 \\ -a\sin(\theta) & a\sin(\phi)\cos(\theta) & a\cos(\theta) \end{vmatrix}.$$

$$= a\cos(\phi)\sin(\theta)(a\cos(\theta)\mathbf{i} + a\sin(\theta)\mathbf{k})$$

$$= a^2\cos(\phi)\sin(\theta)\cos(\theta)\mathbf{i} + a^2\cos(\phi)\sin^2(\theta)\mathbf{k}$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\cos^2(\phi)\sin^2(\theta)\cos^2(\theta) + a^4\cos^2(\phi)\sin^4(\theta)}$$

$$= \sqrt{a^4\cos^2(\phi)\sin^2(\theta)}$$

$$= a^2\cos(\phi)\sqrt{\sin^2(\theta)}$$

$$\text{since } \sqrt{\cos^2(\phi)} = \cos(\phi) \text{ For } \frac{-\pi}{2} \le \phi \le \frac{\pi}{2}$$

Note that

$$\sqrt{\sin^2(\theta)} = \begin{cases} \sin(\theta) & \text{if } 0 \leq \theta \leq \pi \\ -\sin(\theta) & \text{if } \pi \leq \theta \leq 2\pi \end{cases}$$

which implies

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \begin{cases} a^2\cos(\phi)\sin(\theta) & \text{if } 0 \leq \theta \leq \pi \\ -a^2\cos(\phi)\sin(\theta) & \text{if } \pi \leq \theta \leq 2\pi \end{cases}$$

Separating the region into two with $D = D_1 \cup D_2$

$$D_1 = \left\{(\theta, \phi) \,\middle|\, 0 \leq \theta \leq \pi, -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\right\}$$

$$D_2 = \left\{(\theta, \phi) \,\middle|\, \pi \leq \theta \leq 2\pi, -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\right\}$$

$$\text{Surface area} = \iint\limits_{D_1} |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, d\theta d\phi + \iint\limits_{D_2} |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, d\theta d\phi$$

$$= \int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos(\phi)\sin(\theta)d\phi d\theta - \int_\pi^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos(\phi)\sin(\theta)d\phi d\theta$$

$$= a^2\left(\int_0^\pi \sin(\theta)d\theta\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\phi)d\phi\right) - a^2\left(\int_\pi^{2\pi} \sin(\theta)d\theta\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\phi)d\phi\right)$$

$$= a^2\left[-\cos(\theta)\right]_0^\pi \left[\sin(\phi)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - a^2\left[-\cos(\theta)\right]_\pi^{2\pi} \left[\sin(\phi)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4a^2 + 4a^2 = 8a^2$$