DEPARTMENT OF MATHEMATICS

MATH2000
Eigenvalues and diagonalisation

---

(1) Find matrices which diagonalize the following matrices:

(a) $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(2) Diagonalize the matrix $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$.

(3) Which of the following matrices can be diagonalized?

(a) $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 & -1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **(c)** $\begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}$.

(4) Let $C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

1. Find the eigenvalues and eigenvectors of the matrix C.
2. Construct a matrix P such that $P^TCP = D$ where D is a diagonal matrix and find a general expression for the matrix C^n.

(5) (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$.

(b) The differential equations governing the path of a small particle in a rotating fluid field are $\dot{x}_1 = -\omega x_2$ & $\dot{x}_2 = \omega x_1$. Write this as a matrix equation and seek solutions of the form $\boldsymbol{x}(t) = \boldsymbol{z}e^{\lambda t}$ where $\boldsymbol{z}$ is a constant vector. Show that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_0 \cos \omega t \\ x_0 \sin \omega t \end{bmatrix} \text{ if } \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

(6) Solve the system of differential equations

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 + 4x_3 \\ \dot{x}_2 &= 3x_1 + 2x_2 - x_3 \\ \dot{x}_3 &= 2x_1 + x_2 - x_3. \end{aligned}$$

Note that the coefficient matrix here is the same as the matrix in a prior question.

(7) Solve the recurrence relation $x_{n+1} = x_n + 2x_{n-1}$, given that $x_0 = 1$ and $x_1 = 3$.

(8) Find an orthogonal matrix which diagonalizes the matrix $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(9) Let $A = \begin{pmatrix} 2 & 3 & 6 \\ 0 & 5 & 12 \\ 0 & 0 & -1 \end{pmatrix}$.

(a) Find a non-singular matrix P which diagonalizes A.

Background for (b): If an $n \times n$ matrix A has n linearly independent eigenvectors, we have seen that $A = PDP^{-1}$, where D is a diagonal matrix of eigenvalues. We can write

$$D = \lambda_1 E_{11} + \lambda_2 E_{22} + \ldots + \lambda_n E_{nn},$$

where, for example, in the 3×3 case

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then have $A = \lambda_1 S_1 + \lambda_2 S_2 + \ldots + \lambda_n S_n$, where $S_i = PE_{ii}P^{-1}$. This expansion is called the *spectral decomposition* of A.

(b) Use your answer in part (a) to calculate the matrices S_1, S_2 and S_3 for A, and hence write down the spectral decomposition of A.

Background for (c): Since there are three linearly independent eigenvectors for A, they form a basis of $\mathbb{R}^3$. The matrix S_i *projects* onto the subspace of $\mathbb{R}^3$ spanned by the eigenvector $\boldsymbol{v}_i$ corresponding to eigenvalue λ_i. This means that $S_i\boldsymbol{v}_i = \boldsymbol{v}_i$ and $S_i\boldsymbol{v}_j = \boldsymbol{0}$ (for $i \neq j$). In other words, if you take any vector

$$\boldsymbol{w} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3$$

in $\mathbb{R}^3$, multiplying by S_i extracts the $\boldsymbol{v}_i$ component. Using the above expansion of $\boldsymbol{w}$, we have $S_i\boldsymbol{w} = a_i\boldsymbol{v}_i$. We call S_i a *projection matrix.* It turns out that a matrix S is a projection if and only if $S^2 = S$, making them very easy to identify.

(c) Verify that the matrices S_1, S_2 and S_3 obtained in part (b) are indeed projection matrices, then use this fact to write the vector $\boldsymbol{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as a linear combination of eigenvectors of A.