DEPARTMENT OF MATHEMATICS

MATH2000
Eigenvalues and diagonalisation (solutions)

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(1) (a) First solve $\det(\mathrm{A} - \lambda \mathrm{I}) = 0$ to find the eigenvalues.

$$\begin{aligned}
\det(\mathrm{A} - \lambda\mathrm{I}) &= (1-\lambda)[(2-\lambda)(-1-\lambda)+1] + [3(-1-\lambda)+2] + 4[3-2(2-\lambda)] \\
&= (1-\lambda)[\lambda^2-\lambda-1] + [-3\lambda-1] + 4[-1+2\lambda] \\
&= (1-\lambda)[\lambda^2-\lambda-1] + 5\lambda - 5 \\
&= (1-\lambda)[\lambda^2-\lambda-1] - 5(1-\lambda) \\
&= (1-\lambda)[\lambda^2-\lambda-1-5] \\
&= (1-\lambda)[\lambda^2-\lambda-6] \\
&= (1-\lambda)(\lambda+2)(\lambda-3)
\end{aligned}$$

so the eigenvalues are $\lambda = 1, -2, 3$.
If $\mathbf{v}$ is an eigenvector corresponding to $\lambda = 1$, $(A - I)\mathbf{v} = \mathbf{0}$; i.e.

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives the system of equations $-v_2+4v_3 = 0$, $3v_1+v_2-v_3 = 0$, $2v_1+v_2-2v_3 = 0$, which has solution $v_1 = -v_3, v_2 = 4v_3$ so

$$\mathbf{v} = \alpha \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

for $\alpha \neq 0$.
Similarly $\alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha \neq 0$, are the eigenvectors corresponding to $\lambda = -2$, and $\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha \neq 0$, are the eigenvectors corresponding to $\lambda = 3$.
We may choose $\mathbf{v_1} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. These vectors are easily confirmed to be linearly independent (as they theoretically must be, since they correspond to distinct eigenvalues).

Hence $P = \begin{pmatrix} -1 & -1 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ diagonalizes A and

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) First solve $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$ to find the eigenvalues.
$\det(\mathrm{A}-\lambda\mathrm{I}) = (5-\lambda)[(3-\lambda)^2-4] = (5-\lambda)^2(1-\lambda)$ so there are only two eigenvalues, $\lambda = 5, 1$.
As in part (a), if $\mathbf{v}$ is an eigenvector corresponding to $\lambda = 1$,

$$\mathbf{v} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

for $\alpha \neq 0$.
If $\mathbf{v}$ is an eigenvector corresponding to $\lambda = 5$, $(A - 5I)\mathbf{v} = \mathbf{0}$.

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $v_1 + v_2 = 0$.

$$\mathbf{v} = \begin{pmatrix} \alpha \\ -\alpha \\ \beta \end{pmatrix}$$

where α, β are not both zero.
We may choose $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. These vectors are easily checked to be linearly independent.
Hence $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ diagonalizes A and

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(There are many other solutions).

(2) First solve $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$ to find the eigenvalues.
$\det(\mathrm{A}-\lambda\mathrm{I}) = (5-\lambda)[(1-\lambda)(-11-\lambda)+4\cdot 8]-8[4(-11-\lambda)+4\cdot 8]+16[-16+4(1-\lambda)] = 9 - 3\lambda - 5\lambda^2 - \lambda^3$. Note that the coefficients add to zero, so $\lambda = 1$ in the cubic equation

gives zero, hence $(1-\lambda)$ must be a factor. Expanding $(1-\lambda)(a+b\lambda+c\lambda^2)$ then gives $a+(b-a)\lambda+(c-b)\lambda^2-c\lambda^3$, from which it is not too dificult to work out $a=9$, $b=6$ and $c=1$. The characteristic polynomial then factorises as $(1-\lambda)(\lambda+3)^2$. The eigenvalues are $\lambda=-3,1$.

A similar argument to those of the preceding questions shows that if $\mathbf{v}$ is an eigenvector corresponding to $\lambda=1$,

$$\mathbf{v}=\alpha\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

for $\alpha\neq 0$.

If $\mathbf{v}$ is an eigenvector corresponding to $\lambda=-3$, $(A+3I)\mathbf{v}=\mathbf{0}$.

$$A=\begin{pmatrix}8&8&16\\4&4&8\\-4&-4&-8\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Hence $v_1+v_2+2v_3=0$.

$$\mathbf{v}=\begin{pmatrix}-2\alpha-\beta\\\beta\\\alpha\end{pmatrix}$$

where α,β are not both zero.

We may choose the two linearly independent vectors $\mathbf{v_1}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}-2\\0\\1\end{pmatrix}$, which with $\mathbf{v}_3=\begin{pmatrix}2\\1\\-1\end{pmatrix}$ provide 3 linearly indendent eigenvectors.

Hence $P=\begin{pmatrix}-1&-2&2\\1&0&1\\0&1&-1\end{pmatrix}$ diagonalizes A and

$$P^{-1}AP=\begin{pmatrix}-3&0&0\\0&-3&0\\0&0&1\end{pmatrix}.$$

(There are many other solutions).

(3) **(a)** First solve $\det(\mathrm{A}-\lambda\mathrm{I})=0$ to find the eigenvalues.
$\det(\mathrm{A}-\lambda\mathrm{I})=(4-\lambda)[(2-\lambda)^2-3]$ so there are three distinct eigenvalues, $\lambda=4,2\pm\sqrt{3}$. So the matrix is diagonalizable.

(b) $\det(\mathrm{A}-\lambda\mathrm{I})=(4-\lambda)[(2-\lambda)^2-1]-[-(2-\lambda)-2]=(4-\lambda)(2-\lambda)^2$. So the eigenvalues are $\lambda=4,2$. To have 3 linearly independent vectors, we would need 2 corresponding to $\lambda = 2$. If $\mathbf{v}$ is an eigenvector corresponding to $\lambda = 2$, $(A - 2I)\mathbf{v} = \mathbf{0}$, i.e.

$$\begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $v_2 = 0$ and $v_1 + v_3 = 0$, so $\mathbf{v} = \begin{pmatrix} \alpha \\ 0 \\ -\alpha \end{pmatrix}$. But there are not 2 linearly independent vectors of this form. So the matrix is not diagonalizable.

Note that the algebraic multiplicity of $\lambda = 2$ is 2, but the geometric multiplicity is only 1.

(c) This matrix is real symmetric, and thus (from lectures) diagonalizable (by an othogonal matrix).

(4) To find the eigenvalues, we solve the equation $\det(C - I\lambda) = 0$.

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)((2-\lambda)^2 - 1) = 0$$

$$\Rightarrow (4-\lambda)(2-\lambda-1)(2-\lambda+1) = 0$$

So the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 3$, and $\lambda_3 = 1$.

For $\lambda = 4$: $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{v}_1 = 0$

$$\Rightarrow \boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3$: $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{v}_2 = 0$

$$\Rightarrow \boldsymbol{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

For $\lambda = 1$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{v}_3 = 0$

$$\Rightarrow \boldsymbol{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

The matrix P is formed by the eigenvectors:

$$P = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

Now, since C is symmetric, $P^{-1} = P^T$ and we have $P^TCP = D$ or $C = PDP^T$ so $C^n = PD^nP^T$ giving

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 4^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 1^n \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 3^n & 1^n \\ 0 & 3^n & -1^n \\ \sqrt{2}4^n & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3^n + 1^n & 3^n - 1^n & 0 \\ 3^n - 1^n & 3^n + 1^n & 0 \\ 0 & 0 & 2.4^n \end{bmatrix}$$

$$C^n = \begin{bmatrix} \frac{3^n+1}{2} & \frac{3^n-1}{2} & 0 \\ \frac{3^n-1}{2} & \frac{3^n+1}{2} & 0 \\ 0 & 0 & 4^n \end{bmatrix}$$

(5) (a) The eigenvalues are given by $|A - I\lambda| = 0$. So:

$$\begin{vmatrix} -\lambda & \omega \\ -\omega & -\lambda \end{vmatrix}$$

$$\Rightarrow \lambda^2 + \omega^2 = 0$$

$$\Rightarrow \lambda = \pm i\omega$$

For $\lambda_1 = i\omega$, $\begin{bmatrix} -i\omega & \omega \\ -\omega & -i\omega \end{bmatrix} \boldsymbol{v}_1 = 0$

$$\Rightarrow \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

For $\lambda_1 = -i\omega$, $\begin{bmatrix} i\omega & \omega \\ -\omega & i\omega \end{bmatrix} \boldsymbol{v}_2 = 0$

$$\Rightarrow \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

(b) Substition of $\boldsymbol{x}(t) = \boldsymbol{z}e^{\lambda t}$ gives

$$\lambda \boldsymbol{z} e^{\lambda t} = A\boldsymbol{z}e^{\lambda t} \text{ or } \lambda\boldsymbol{z} = A\boldsymbol{z}$$

So λ and $\boldsymbol{z}$ are eigenpairs of the matrix A:

$$\lambda_1 = i\omega, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \lambda_2 = -i\omega, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

so the general solutions is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{i\omega t} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-i\omega t}$$

where c_1 and c_2 are in general complex.
Since $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = c_2 = \frac{x_0}{2}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_0 \frac{e^{i\omega t} + e^{-i\omega t}}{2} \\ x_0 \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_0 \cos \omega t \\ x_0 \sin \omega t \end{bmatrix}$$

Note that we can rewrite the system of two first order ODEs as one second order ODE.

$$\begin{aligned} \dot{x}_1 = -\omega x_2 \quad &\Rightarrow \quad \ddot{x}_1 = -\omega \dot{x}_2 \\ &\Rightarrow \quad \ddot{x}_1 = -\omega(\omega x_1) \\ &\Rightarrow \quad \ddot{x}_1 + \omega^2 x_1 = 0. \end{aligned}$$

Our solution for x_1 and x_2 is consistent with our usual method of solution for a linear homogeneous second order ODE.

(6) Write the system in matrix form: $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

From an earlier question it can be shown, $P = \begin{pmatrix} -1 & -1 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ diagonalizes A and

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D$$

say. Let $\mathbf{x} = P\mathbf{y}$ so $P\dot{\mathbf{y}} = AP\mathbf{y} \Rightarrow \dot{\mathbf{y}} = P^{-1}AP\mathbf{y} = D\mathbf{y}$, which corresponds to the system of equations $\dot{y_1} = y_1,\ \dot{y_2} = -2y_2,\ \dot{y_3} = 3y_3$. The solution is $\mathbf{y} = \begin{pmatrix} c_1e^t \\ c_2e^{-2t} \\ c_3e^{3t} \end{pmatrix}$. Hence

$$
\begin{aligned}
\mathbf{x} = P\mathbf{y} & \\
&= \begin{pmatrix} -1 & -1 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1e^t \\ c_2e^{-2t} \\ c_3e^{3t} \end{pmatrix} \\
&= \begin{pmatrix} -c_1e^t - c_2e^{-2t} + c_3e^{3t} \\ 4c_1e^t + c_2e^{-2t} + 2c_3e^{3t} \\ c_1e^t + c_2e^{-2t} + c_3e^{3t} \end{pmatrix}.
\end{aligned}
$$

(7) Consider the system $\begin{cases} x_{n+1} = x_n + 2x_{n-1} \\ x_n = x_n \end{cases}$. In matrix form, $\mathbf{x_n} = A\mathbf{x_{n-1}}$ where $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{x_n} = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$. The solution is $\mathbf{x_n} = A^n\mathbf{x_0}$, and here $\mathbf{x_0} = \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

We will diagonalize A in order to find A^n. First we solve $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$ to find the eigenvalues.

$\det(\mathrm{A} - \lambda\mathrm{I}) = (1-\lambda)(-\lambda) - 2 = (\lambda - 2)(\lambda + 1)$ so the eigenvalues are $\lambda = 2, -1$. The eigenvectors corresponding to 2 are $\begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix}$ for $\alpha \neq 0$ and the eigenvectors corresponding to -1 are $\begin{pmatrix} -\alpha \\ \alpha \end{pmatrix}$ for $\alpha \neq 0$. Hence $P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ diagonalizes A to $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

Now $D = P^{-1}AP \Rightarrow D^n = P^{-1}A^nP \Rightarrow PD^nP^{-1} = A^n$ i.e.

$$
\begin{aligned}
A^n &= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 2^{n+1} - (-1)^{n+1} & 2^{n+1} + 2(-1)^{n+1} \\ 2^n - (-1)^n & 2^n + 2(-1)^n \end{pmatrix}.
\end{aligned}
$$

Hence

$$
\mathbf{x_n} = A^n\mathbf{x_0} = A^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 \times 2^{n+1} - (-1)^{n+1} \\ 4 \times 2^n - (-1)^n \end{pmatrix}.
$$

Since $\mathbf{x_n} = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$, the solution is $x_n = \frac{1}{3}(2^{n+2} - (-1)^n)$.

(8) There are only two eigenvalues, $\lambda = 5, 1$.

$$
\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
$$

is an eigenvector corresponding to $\lambda = 1$. For $\lambda = 5$ we may choose $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. These vectors are easily checked to be linearly independent.

Hence $P = [\hat{\mathbf{v}}_1|\hat{\mathbf{v}}_2|\hat{\mathbf{v}}_3] = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ diagonalizes A and $P^T = P^{-1}$ so

$$P^T AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(9) Part (a)

First determine the eigenvalues of the matrix A. Solving the characteristic equation.

$\det(A) = det \begin{pmatrix} 2-\lambda & 3 & 6 \\ 0 & 5-\lambda & 12 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (2-\lambda)(5-\lambda)(-1-\lambda) + 0 + 0 = 0$

The eignvalues obtained are $\lambda_1 = 2$ $\lambda_2 = 5$ and $\lambda_3 = -1$.

Since the eigenvalues λ_i are distinct and non zero, this implies can use a matrix of eigenvectors $P = [v_1, v_2, v_3]$ to diagonalize A.

First determine the eigenvectors

For $\lambda_1 = 2$

$$\begin{pmatrix} 2-2 & 3 & 6 \\ 0 & 5-2 & 12 \\ 0 & 0 & -1-2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 6 \\ 0 & 3 & 12 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Setting $a = 1$

$v_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_2 = 5$

$$\begin{pmatrix} -3 & 3 & 6 \\ 0 & 0 & 12 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies -3a + 3b = 0 \implies a = b \text{ Setting } b = 1$$

So $v_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = -1$

$$\begin{pmatrix} 3 & 3 & 6 \\ 0 & 6 & 12 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies 6b + 12c = 0 \implies -2c = b \text{ Setting } b = 1$$

So $v_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

So $P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. This can be easily inverted to get $P^{-1} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

And $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ a diagonal matrix of the eigenvalues.

Part (b)

$$S_1 = PE_{11}P^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_2 = PE_{22}P^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_3 = PE_{33}P^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

And so the spectral decomposition of A is

$$A = \lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3$$

$$\begin{pmatrix} 2 & 3 & 6 \\ 0 & 5 & 12 \\ 0 & 0 & -1 \end{pmatrix} = 2\begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} - 1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Part (c)

First verify that S_1, S_2, S_3, are projection matrices.

$$S_1S_1 = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_1S_1 = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So S_1 is a projection matrix. Similarly it can be shown that $S_2S_2 = S_2$ and $S_3S_3 = S_3$

Now determining the projection of w on v_1 multiply by the projection matrix S_1

$$S_1 w = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-b-2c \\ 0 \\ 0 \end{pmatrix} = (a-b-2c)v_1$$

Similary

$$S_2 w = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b+2c \\ b+2c \\ 0 \end{pmatrix} = (b+2c)v_2$$

$$S_3 w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -2c \\ c \end{pmatrix} = cv_3$$

And so w as a linear combination of eigenvectors of A is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a-b-2c)v_1 + (b+2c)v_2 + cv_3$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a-b-2c)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (b+2c)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$