DEPARTMENT OF MATHEMATICS

MATH2000
First Order ODEs solutions.

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(1) (a)

The ODE is separable:
The equation can be written in the form

$$y' = x\left(\frac{1-y^2}{y}\right)$$

which is separable. Hence we have

$$\int \frac{y}{1-y^2}dy = \int x\ dx.$$

For the left hand side, substitute $u = y^2 \ \Rightarrow\ du = 2y\ dy$

$$\begin{aligned}
\Rightarrow \int \frac{y}{1-y^2}dy &= \frac{1}{2}\int \frac{du}{1-u} = -\frac{1}{2}\ln|1-u| \\
&= -\frac{1}{2}\ln|1-y^2| = \text{ r.h.s} \\
&= \frac{1}{2}x^2 + c \\
\Rightarrow 1 - y^2 &= ae^{-x^2} \qquad (a = \pm e^{-2c})
\end{aligned}$$

or

$$y^2 = 1 - ae^{-x^2}.$$

(b)

Multiply both sides of the ODE by ye^{x^2} to make it exact:
The equation becomes

$$\begin{aligned}
ye^{x^2}y' + xy^2e^{x^2} &= xe^{x^2} \\
\Rightarrow (y^2-1)xe^{x^2} + ye^{x^2} &= 0.
\end{aligned}$$

We can check this is homogeneous since

$$\frac{\partial}{\partial y}\left((y^2-1)xe^{x^2}\right) = 2xye^{x^2} = \frac{\partial}{\partial y}\left(ye^{x^2}\right).$$

Therefore we seek a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = (y^2-1)xe^{x^2}$$

and

$$\frac{\partial f}{\partial y} = ye^{x^2}.$$

From the first equation, such a function is of the form

$$f(x,y) = \frac{1}{2}(y^2-1)e^{x^2} + g(y)$$

for some function g of y only. Differentiating this partially with respect to y gives

$$\frac{\partial f}{\partial y} = ye^{x^2} + g'(y).$$

Comparing this expression with the other expression for $\dfrac{\partial f}{\partial y}$ above, we must have $g'(y) = 0$ and therefore $g(y)$ is constant. The implicit solution to the ODE is then

$$f(x,y) = \frac{1}{2}(y^2-1)e^{x^2} = c,$$

which can be expressed in the form of the solution to part (a).

(2) To show that the equation

$$P(x,y) + Q(x,y)y' = 0$$

is exact, we show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

In the case of $2x^2 + xy^2 + x^2yy' = 0$, $P(x,y) = 2x^2 + xy^2$ and $Q(x,y) = x^2y$

$$\Rightarrow \quad \frac{\partial P}{\partial y} = 2xy = \frac{\partial Q}{\partial x}$$

so the equation is indeed exact. Therefore there exists a function $f(x,y)$ such that $f_x = P$ and $f_y = Q$ and $f(x,y) = C$ implicitly defines $y(x)$. That is,

$$\begin{aligned} f_x &= 2x^2 + xy^2 \\ \Rightarrow \ f(x,y) &= \frac{2}{3}x^3 + \frac{1}{2}x^2y^2 + g(y) \end{aligned}$$

where $g(y)$ is an arbitrary function of y. Differentiating with respect to y then gives

$$f_y = x^2y + g'(y).$$

We also have

$$\begin{aligned} Q(x,y) &= x^2y = f_y \\ \Rightarrow \ g'(y) &= 0 \end{aligned}$$

so $g(y)$ must be a constant. Therefore

$$\frac{2}{3}x^3 + \frac{1}{2}x^2y^2 = c$$

implicitly defines our general solution $y(x)$. Imposing the condition $y(1) = -2$ gives $c = 8/3$. Therefore

$$\begin{aligned} y^2 &= \frac{16}{3x^2} - \frac{4x}{3} \\ \Rightarrow y &= -\sqrt{\frac{16}{3x^2} - \frac{4x}{3}} \end{aligned}$$

where we chose the minus sign so that the solution satisfies $y(1) = -2$.

(3) You should recognise that the equation is separable. Therefore

$$\int \frac{dy}{y} = 2\int \frac{x-1}{x(x-2)}dx.$$

For the right hand side, we can use partial fractions. You should obtain

$$\frac{x-1}{x(x-2)} = \frac{1/2}{x} + \frac{1/2}{x-2}.$$

Therefore

$$\begin{aligned} 2\int \frac{x-1}{x(x-2)}dx &= \int dxx + \int dxx - 2 \\ &= \ln|x| + \ln|x-2| + c \\ &= \text{l.h.s } = \ln|y| \\ \Rightarrow y &= ax(x-2) \qquad (a = \pm e^c) \end{aligned}$$

is the general solution.

Now consider $y(x_0) = y_0$.

(a) Note that when $x = 0$ or $x = 2$, the general solution becomes $y = 0$. Therefore if $x_0 = 0$ or $x_0 = 2$ *and* $y_0 \neq 0$, there cannot be a solution. Note also in this case the ODE is not even satisfied by these initial conditions. So, no solution for

$$(x_0, y_0) = (0, p),\ (2, p) \quad p \neq 0.$$

(b) If $x_0 = 0$ or $x_0 = 2$ and $y_0 = 0$, the general solution is consistent for *any* value of the constant a in the general solution. So in the case

$$(x_0, y_0) = (0, 0),\ (2, 0)$$

there are infinitely many solutions.

(c) For all values of $x_0 \neq 0, 2$ we are able to solve for the constant a in the general solution, which will ensure there is precisely one solution. So, there is a unique solution for

$$(x_0, y_0) = (p, q), \quad p \neq 0, 2 \text{ and } \forall q.$$

Note that we can write the ODE as

$$y' = \frac{2(x-1)y}{x(x-2)} = f(x, y).$$

In the case when $x = 0, 2$, the function $f(x, y)$ is not defined, so is certainly not continuous there. The criterion for existence as presented in lectures then does not give any information about existence of solutions for $x = 0, 2$. When $x \neq 0, 2$, $f(x, y)$ is continuous so we can conclude that there exists at least one solution (existence). Also,

$$f_y = \frac{2(x-1)}{x(x-2)},$$

which is also continuous when $x \neq 0, 2$ so we can conclude that there exists at most one solution (uniqueness).

(4) Set $P(x, y) = ax + by$ and $Q(x, y) = cx + dy$. Recall from lectures that an equation of the form

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0$$

is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

In this case

$$\frac{\partial P}{\partial y} = b, \qquad \frac{\partial Q}{\partial x} = c,$$

so the equation is exact if and only if $b = c$. Imposing this condition gives the exact equation

$$ax + by + (bx + dy)\frac{dy}{dx} = 0.$$

Hence there is a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = ax + by, \qquad \frac{\partial f}{\partial y} = bx + dy.$$

Integrating the first equation (treating y as constant) gives

$$\begin{aligned} f(x, y) &= \frac{1}{2}ax^2 + bxy + g(y) \qquad \text{(some function } g(y)) \\ \Rightarrow \frac{\partial f}{\partial y} &= bx + g'(y) \\ &= bx + dy \end{aligned}$$

by the second equation above. Therefore

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\begin{aligned}
g'(y) &= dy \\
\Rightarrow \; g(y) &= \frac{1}{2}dy^2 \qquad \text{(ignoring constant which appears in next line)} \\
\Rightarrow \; f(x,y) &= \frac{1}{2}ax^2 + bxy + \frac{1}{2}dy^2 = k \qquad (k \text{ constant})
\end{aligned}
$$

is an implicit solution to the ODE.