DEPARTMENT OF MATHEMATICS

MATH2000
Flux integrals and Gauss' divergence theorem (solutions)

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(1) The hemisphere can be represented as

$$V = \{(r, \theta, \phi) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/2\}.$$

We have by direct calculation

$$\mathrm{div}\boldsymbol{F} = 3(x^2 + y^2 + z^2) = 3r^2$$

in terms of spherical coordinates.

$$\begin{aligned}
\Rightarrow \iiint\limits_V \mathrm{div}\boldsymbol{F}\ dV &= 3\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 r^2 \cdot r^2 \sin\phi\ dr\ d\theta\ d\phi \\
&= 3\left(\int_0^{\pi/2} \sin\phi\ d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r^4\ dr\right) \\
&= 3 \times 1 \times 2\pi \times \frac{1}{5} = \frac{6\pi}{5}.
\end{aligned}$$

Now to evaluate $\oiint\limits_S \boldsymbol{F} \cdot \boldsymbol{n}\ dS$. In this case the surface comprises of two parts: the base of the hemisphere which lies in the x-y plane, denoted S_1, and the part of the sphere itself, denoted S_2. So that

$$\oiint\limits_S \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint\limits_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}_1\ dS + \iint\limits_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}_2\ dS$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are outwardly pointing unit normal vectors to the surfaces S_1 and S_2 respectively.

We expect the integral $\iint\limits_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}_1\ dS$ to be zero since the $\boldsymbol{k}$ component of $\boldsymbol{F}$ is 0 when $\boldsymbol{F}$ is restricted to the x-y plane, so there is no flux across that surface.

To verify this by direct calculation, we must first parametrise the surface S_1. Since it is just a circular disc in the x-y plane, we have

$$\boldsymbol{r}(r, \theta) = r\cos\theta\boldsymbol{i} + r\sin\theta\boldsymbol{j}, \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

We then take the tangent vectors

$$\begin{aligned}
\boldsymbol{r}_\theta &= -r\sin\theta\boldsymbol{i} + r\cos\theta\boldsymbol{j} \\
\boldsymbol{r}_r &= \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}
\end{aligned}$$

We can calculate $\boldsymbol{r}_r \times \boldsymbol{r}_\theta = r\boldsymbol{k}$. However, this is directed into the solid. We require this vector to be directed outwards from the solid, so instead we'll take $\boldsymbol{r}_\theta \times \boldsymbol{r}_r = -r\boldsymbol{k}$.

In terms of our parametrisation,

$$\boldsymbol{F}(r,\theta) = r^3\cos^3\theta\boldsymbol{i} + r^3\sin^3\theta\boldsymbol{j},$$

so that the dot product

$$\boldsymbol{F}(r,\theta)\cdot(\boldsymbol{r}_\theta \times \boldsymbol{r}_r) = 0$$

which tells us that the flux across S_1 will be zero as originally thought.

To calculate the flux across S_2, we parametrise S_2 (compare with the spherical coordinate transformation) as

$$\boldsymbol{r}(\theta,\phi) = \cos\theta\sin\phi\boldsymbol{i} + \sin\theta\sin\phi\boldsymbol{j} + \cos\phi\boldsymbol{k}, \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/2.$$

The tangent vectors are

$$\begin{aligned}\boldsymbol{r}_\theta &= -\sin\theta\sin\phi\boldsymbol{i} + \cos\theta\sin\phi\boldsymbol{j}\\ \boldsymbol{r}_\phi &= \cos\theta\cos\phi\boldsymbol{i} + \sin\theta\cos\phi\boldsymbol{j} - \sin\phi\boldsymbol{k}.\end{aligned}$$

To find a vector normal to the surface S_2, take

$$\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = \cos\theta\sin^2\phi\boldsymbol{i} + \sin\theta\sin^2\phi\boldsymbol{j} + \sin\phi\cos\phi\boldsymbol{k}.$$

We should check the direction to make sure it is directed outwards from the surface. Take for example the parameter values $\phi = \pi/2$ and $\theta = 0$. This gives $\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = \boldsymbol{i}$ which is directed out, so the direction is ok.

In terms of the parameters, we can write

$$\boldsymbol{F}(\theta,\phi) = \cos^3\theta\sin^3\phi\boldsymbol{i} + \sin^3\theta\sin^3\phi\boldsymbol{j} + \cos^3\phi\boldsymbol{k}.$$

so that the dot product

$$\boldsymbol{F}(\theta,\phi)\cdot(\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta) = \cos^4\theta\sin^5\phi + \sin^4\theta\sin^5\phi + \sin\phi\cos^4\phi.$$

The flux integral we need to evaluate is then

$$\int_0^{2\pi}\int_0^{\pi/2}\left((\cos^4\theta + \sin^4\theta)\sin^5\phi + \sin\phi\cos^4\phi\right)\, d\phi\, d\theta.$$

Using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ we have

$$\begin{aligned}\cos^4\theta = \frac{1}{4}(1+\cos 2\theta)(1+\cos 2\theta) &= \frac{1}{4}(1+2\cos 2\theta + \cos^2 2\theta)\\ &= \frac{1}{4}(1+2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta))\\ &= \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.\end{aligned}$$

It follows that

$$\begin{aligned}
\cos^4\theta + \sin^4\theta &= \cos^4\theta + (1-\cos^2\theta)(1-\cos^2\theta) \\
&= 1 - 2\cos^2\theta + 2\cos^4\theta \\
&= 1 - (1+\cos 2\theta) + \frac{3}{4} + \cos 2\theta + \frac{1}{4}\cos 4\theta \\
&= \frac{3}{4} + \frac{1}{4}\cos 4\theta.
\end{aligned}$$

Finally note that we can write $\sin^5\phi = \sin\phi(1 - 2\cos^2\phi + \cos^4\phi)$.

Putting this together, the flux across S_2 is

$$\begin{aligned}
& \int_0^{2\pi}\int_0^{\pi/2}\left(\frac{3}{4} + \frac{1}{4}\cos 4\theta\right)\sin\phi(1 - 2\cos^2\phi + \cos^4\phi)\ d\phi\ d\theta \\
& + \int_0^{2\pi}\int_0^{\pi/2} \sin\phi\cos^4\phi\ d\phi\ d\theta \\
= & \left(\int_0^{2\pi}\left(\frac{3}{4} + \frac{1}{4}\cos 4\theta\right) d\theta\right)\left(\int_0^{\pi/2}\sin\phi(1 - 2\cos^2\phi + \cos^4\phi)d\phi\right) \\
& + \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2}\sin\phi\cos^4\phi\ d\phi\right)
\end{aligned}$$

Using the substitution $u = \cos\phi$ in both ϕ integrals gives

$$\begin{aligned}
= & \left(\left[\frac{3}{4}\theta + \frac{1}{16}\sin 4\theta\right]_0^{2\pi} \times \int_0^1 (1 - 2u^2 + u^4)du\right) + \left(2\pi \times \int_0^1 u^4\ du\right) \\
= & \frac{3\pi}{2}\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 + 2\pi\left[\frac{1}{5}u^5\right]_0^1 \\
= & \frac{3\pi}{2}\left(1 - \frac{2}{3} + \frac{1}{5}\right) + \frac{2\pi}{5} = \frac{6\pi}{5}.
\end{aligned}$$

Therefore

$$\oiint_S \boldsymbol{F}\cdot\boldsymbol{n}\ dS = \iint_{S_1}\boldsymbol{F}\cdot\boldsymbol{n}_1\ dS + \iint_{S_2}\boldsymbol{F}\cdot\boldsymbol{n}_2\ dS = 0 + \frac{6\pi}{5} = \frac{6\pi}{5}.$$

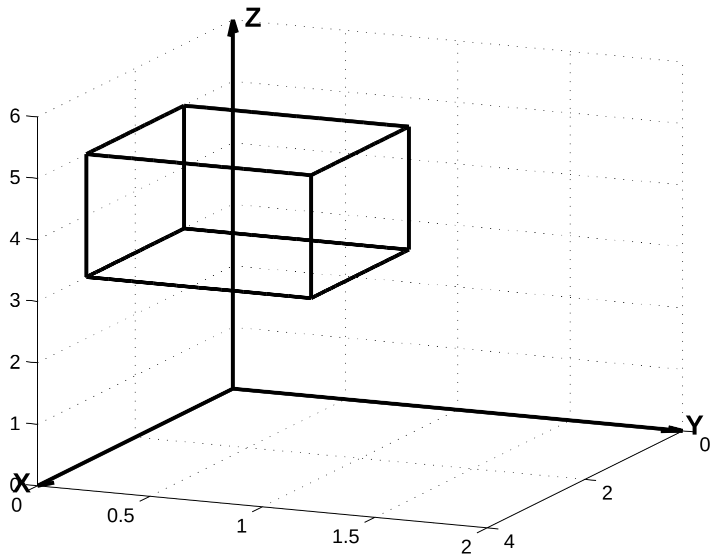
So we have shown that for this example both sides of the equation in Gauss' theorem are equal.

(2) Note that in this case we *cannot* use Gauss' divergence theorem since the vector field $\boldsymbol{F} = \frac{1}{x}\boldsymbol{i}$ is undefined at any point in the y-z plane (ie. when $x = 0$), part of which lies in the region enclosed by the surface. We must evaluate $\oiint_S \boldsymbol{F} \cdot \boldsymbol{n} \, dS$ directly.

Since the surface is the unit sphere, the position vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ will also be an outwardly pointing unit normal (since $x^2 + y^2 + z^2 = 1$ on the surface). Taking $\boldsymbol{n} = \boldsymbol{r}$, we have that $\boldsymbol{F} \cdot \boldsymbol{n} = 1$. Therefore the flux evaluates to

$$\begin{aligned} \oiint_S \boldsymbol{F} \cdot \boldsymbol{n} \, dS &= \oiint_S dS \\ &= \text{surface area of the unit sphere} \\ &= 4\pi. \end{aligned}$$

(3) A diagram of the solid is as follows:



The outward flux can be calculated as

$$\oiint_S \boldsymbol{F} \cdot \boldsymbol{n}$$

where S is the closed surface of the box, $\boldsymbol{F}$ is the vector field, and $\boldsymbol{n}$ is an outwardly pointing unit normal vector. The surface S consists of six open surfaces: the six faces of the box. We can evaluate the flux integral directly by calculating the outward flux through each face:

$$\oiint_S \boldsymbol{F} \cdot \boldsymbol{n} = \iint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n} \, dS + \iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n} \, dS + \iint_{S_3} \boldsymbol{F} \cdot \boldsymbol{n} \, dS + \iint_{S_4} \boldsymbol{F} \cdot \boldsymbol{n} \, dS + \iint_{S_5} \boldsymbol{F} \cdot \boldsymbol{n} \, dS + \iint_{S_6} \boldsymbol{F} \cdot \boldsymbol{n} \, dS.$$

We represent each open surface follows:

- S_1 is the "base" of the box which lies in the plane $z = 3$ (and is therefore parallel to the x-y plane). An outwardly pointing unit normal is $\boldsymbol{n} = -\boldsymbol{k}$. Restricted to S_1, the vector field is given by

$$\boldsymbol{F} = x\boldsymbol{i} + 12y\boldsymbol{j} + 9\boldsymbol{k}, \text{ for } 1 \leq x \leq 3,\ 0 \leq y \leq 1.$$

  Therefore over S_1,

$$\boldsymbol{F} \cdot \boldsymbol{n} = (x\boldsymbol{i} + 12y\boldsymbol{j} + 9\boldsymbol{k}) \cdot (-\boldsymbol{k}) = -9.$$

  The surface integral is then

$$\iint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint_{S_1} (-9)\ dS = -9 \iint_{S_1} dS.$$

  Since $\iint_{S_1} dS$ is just the area of S_1, which is a rectangle of area $= 2$, so

$$\iint_{S_1} dS = 2 \ \Rightarrow\ \iint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = -9 \times 2 = -18.$$

- S_2 is the "lid" of the box which lies in the plane $z = 5$. An outwardly pointing unit normal is $\boldsymbol{n} = \boldsymbol{k}$. Restricted to S_2, the vector field is given by

$$\boldsymbol{F} = x\boldsymbol{i} + 12y\boldsymbol{j} + 15\boldsymbol{k}, \text{ for } 1 \leq x \leq 3,\ 0 \leq y \leq 1.$$

  Therefore over S_2,

$$\boldsymbol{F} \cdot \boldsymbol{n} = (x\boldsymbol{i} + 12y\boldsymbol{j} + 15\boldsymbol{k}) \cdot \boldsymbol{k} = 15.$$

  The surface integral is then

$$\iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint_{S_2} (15)\ dS = 25 \iint_{S_2} dS.$$

  Since $\iint_{S_2} dS$ is just the area of S_2, which is a rectangle of area $= 2$, so

$$\iint_{S_2} dS = 2 \ \Rightarrow\ \iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = 15 \times 2 = 30.$$

- S_3 is the "back" of the box which lies in the plane $x = 1$. An outwardly pointing unit normal is $\boldsymbol{n} = -\boldsymbol{i}$. Restricted to S_3, the vector field is given by

$$\boldsymbol{F} = 1\boldsymbol{i} + 12y\boldsymbol{j} + 3z\boldsymbol{k}, \text{ for } 0 \leq y \leq 1,\ 3 \leq z \leq 5.$$

Therefore over S_3,

$$\boldsymbol{F} \cdot \boldsymbol{n} = (1\boldsymbol{i} + 12y\boldsymbol{j} + 3z\boldsymbol{k}) \cdot (-\boldsymbol{i}) = -1.$$

The surface integral is then

$$\iint_{S_3} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint_{S_3} (-1)\ dS = -\iint_{S_3} dS.$$

Since $\displaystyle\iint_{S_3} dS$ is just the area of S_3, which is a rectangle of area $= 2$, so

$$\iint_{S_3} dS = 2 \ \Rightarrow\ \iint_{S_3} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = -2.$$

– S_4 is the "front" of the box which lies in the plane $x = 3$. An outwardly pointing unit normal is $\boldsymbol{n} = \boldsymbol{i}$. Restricted to S_4, the vector field is given by

$$\boldsymbol{F} = 3\boldsymbol{i} + 12y\boldsymbol{j} + 3z\boldsymbol{k}, \text{ for } 0 \le y \le 1,\ 3 \le z \le 5.$$

Therefore over S_4,

$$\boldsymbol{F} \cdot \boldsymbol{n} = (3\boldsymbol{i} + 12y\boldsymbol{j} + 3z\boldsymbol{k}) \cdot (\boldsymbol{i}) = 3.$$

The surface integral is then

$$\iint_{S_4} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint_{S_4} (3)\ dS = 3\iint_{S_4} dS.$$

Since $\displaystyle\iint_{S_4} dS$ is just the area of S_4, which is a rectangle of area $= 2$, so

$$\iint_{S_4} dS = 2 \ \Rightarrow\ \iint_{S_4} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = 3 \times 2 = 6.$$

– S_5 is the 'left side" of the box which lies in the plane $y = 0$ (the x-z plane). An outwardly pointing unit normal is $\boldsymbol{n} = -\boldsymbol{j}$. Restricted to S_5, the vector field is given by

$$\boldsymbol{F} = x\boldsymbol{i} + 3z\boldsymbol{k}, \text{ for } 1 \le x \le 3,\ 3 \le z \le 5.$$

Therefore over S_5,

$$\boldsymbol{F} \cdot \boldsymbol{n} = (x\boldsymbol{i} + 3z\boldsymbol{k}) \cdot (-\boldsymbol{j}) = 0.$$

The surface integral is then

$$\iint_{S_5} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint_{S_5} (0)\ dS = 0.$$

- S_6 is the "right side" of the box which lies in the plane $y = 1$. An outwardly pointing unit normal is $\boldsymbol{n} = \boldsymbol{j}$. Restricted to S_6, the vector field is given by

$$\boldsymbol{F} = x\boldsymbol{i} + 12\boldsymbol{j} + 3z\boldsymbol{k}, \text{ for } 1 \leq x \leq 3,\ 3 \leq z \leq 5.$$

  Therefore over S_6,

$$\boldsymbol{F} \cdot \boldsymbol{n} = (x\boldsymbol{i} + 12\boldsymbol{j} + 3z\boldsymbol{k}) \cdot (\boldsymbol{j}) = 12.$$

  The surface integral is then

$$\iint\limits_{S_6} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint\limits_{S_6} (12)\ dS = 12 \iint\limits_{S_6} dS.$$

  Since $\iint\limits_{S_6} dS$ is just the area of S_6, which is a rectangle of area $= 4$, so

$$\iint\limits_{S_6} dS = 4 \ \Rightarrow \ \iint\limits_{S_6} \boldsymbol{F} \cdot \boldsymbol{n}\ dS = 12 \times 4 = 48.$$

Putting all of this information together gives

$$\oint\!\!\!\oint\limits_{S} \boldsymbol{F} \cdot \boldsymbol{n} = \iint\limits_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}\ dS + \iint\limits_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}\ dS + \iint\limits_{S_3} \boldsymbol{F} \cdot \boldsymbol{n}\ dS + \iint\limits_{S_4} \boldsymbol{F} \cdot \boldsymbol{n}\ dS + \iint\limits_{S_5} \boldsymbol{F} \cdot \boldsymbol{n}\ dS + \iint\limits_{S_6} \boldsymbol{F} \cdot \boldsymbol{n}\ dS$$

$$= -18 + 30 + (-2) + 6 + 0 + 48 = 64.$$

Using the divergence theorem, we can also calculate the outward flux as

$$\iiint\limits_{V} \text{div}\boldsymbol{F}\ dV,$$

where V is the region enclosed by S (ie. the box). We can calculate

$$\text{div}\boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(12y) + \frac{\partial}{\partial z}(3z) = 1 + 12 + 3 = 16.$$

The outward flux is then

$$\begin{aligned} \iiint\limits_{V} \text{div}\boldsymbol{F}\ dV &= 16 \iiint\limits_{V} dV \\ &= 16 \times \ (\text{vol. of box}) \\ &= 16 \times (2 \times 2 \times 1) = 64. \end{aligned}$$

We have therefore verified the divergence theorem. In this case, it is a lot less work to calculate the volume integral compared to the flux integral.

(4) Use the divergence theorem. The region (in this case a sphere of radius 5) can be represented as

$$V = \{(r, \theta, \phi) \mid 0 \le r \le 5,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

in term of spherical polar coordinates. We also have

$$\text{div}\boldsymbol{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(5z) = 3 + 4 + 5 = 12.$$

Hence by the divergence theorem the flux out of the surface is

$$\begin{aligned}
\iiint_V \text{div}\boldsymbol{F}\ dV &= 12 \int_0^{2\pi} \int_0^{\pi} \int_0^5 r^2 \sin\phi\ dr\ d\phi\ d\theta \\
&= 12 \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin\phi\ d\phi \right) \left(\int_0^5 r^2\ dr \right) \\
&= 12 \times 2\pi \times 2 \times \frac{125}{3} \\
&= 2000\pi.
\end{aligned}$$

Alternatively, we could make the observation that

$$\begin{aligned}
\iiint_V \text{div}\boldsymbol{F}\ dV &= 12 \iiint_V dV \\
&= 12 \times (\text{ volume of sphere of radius } 5) \\
&= 12 \times \left(\frac{4}{3}\pi 5^3 \right) = 2000\pi.
\end{aligned}$$

(5) We need to find $\oiint_S \boldsymbol{F} \cdot \mathbf{n}\ dS$. By Gauss' divergence theorem this is equal to $\iiint_V \text{div}\boldsymbol{F}\ dV$.

$$\begin{aligned}
\text{div}\boldsymbol{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(6z) \\
&= 1 + 3 + 6 = 10.
\end{aligned}$$

In cylindrical polar coordinates, the cone is $z^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2 \Rightarrow z = r$ in this case since $0 \le z \le 2$. The region in R^3 is

$$V = \{(r, \theta, z) \mid 0 \le z \le 2, 0 \le \theta \le 2\pi, 0 \le r \le z\}.$$

So flux

$$
\begin{aligned}
&= 10\int_0^2\int_0^{2\pi}\int_0^z r\ dr\ d\theta\ dz \\
&= 10\int_0^2\int_0^{2\pi}\frac{1}{2}z^2\ d\theta\ dz \\
&= 5\int_0^2 z^2 dz\int_0^{2\pi}\ d\theta \\
&= 5\left[\frac{1}{3}z^3\right]_0^2\cdot 2\pi \\
&= \frac{80\pi}{3}.
\end{aligned}
$$

(6) $\boldsymbol{F} = (x^3+xy^2+xz^2)\boldsymbol{i}+(x^2y+y^3+yz^2)\boldsymbol{j}+(x^2z+y^2z+z^3)\boldsymbol{k}$ so

$$
\begin{aligned}
\mathrm{div}\boldsymbol{F} &= \frac{\partial}{\partial x}(x^3+xy^2+xz^2)+\frac{\partial}{\partial y}(x^2y+y^3+yz^2) \\
&\quad +\frac{\partial}{\partial z}(x^2z+y^2z+z^3) \\
&= (3x^2+y^2+z^2)+(x^2+3y^2+z^2) \\
&\quad +(x^2+y^2+3z^2) \\
&= 5r^2.
\end{aligned}
$$

The sphere is described by

$$V = \{(r,\theta,\phi)\mid 0\le r\le a, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}.$$

So by Gauss' divergence theorem, the flux across the surface of the sphere

$$
\begin{aligned}
&= \int_0^a\int_0^{2\pi}\int_0^{\pi}(5r^2)\cdot r^2\sin\phi\ d\phi\ d\theta\ dr \\
&= 5\int_0^a r^4\ dr\ \int_0^{2\pi} d\theta\ \int_0^{\pi}\sin\phi\ d\phi \\
&= 5\left[\frac{1}{5}r^5\right]_0^a\cdot[\theta]_0^{2\pi}\cdot[-\cos\phi]_0^{\pi} \\
&= 5\cdot\frac{1}{5}a^5\cdot 2\pi\cdot 2 \\
&= 4\pi a^5.
\end{aligned}
$$