DEPARTMENT OF MATHEMATICS

MATH2000
Gaussian elimination, LU and PLU decomposition (solutions)

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(1) For each part of this question, there are many possible row equivalent matrices. The following are only a few examples.

(a) $\begin{pmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ by doing $r_1 \leftrightarrow r_2$ (interchange row 1 and row 2).

(b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ by doing $r_2 \to r_2 - r_1, r_3 \to r_3 - r1, r_2 \leftrightarrow r_3$,

or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by doing $r_1 \leftrightarrow r_3, r_2 \to r_2 - r_1, r_3 \to r_3 - r_1, r_3 \to r_3 - r_2$,

or many others.

(c) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ by doing $r_1 \leftrightarrow r_2, r_3 \to r_3 - 2r_1$.

(d) $\begin{pmatrix} 1 & -2 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ by doing $r_2 \to r_2 + 4r_1, r_3 \to r_3 - 7r_1, r_3 \to r_3 + 2r_2$.

(2) One way is to start with $r_2 \to r_2 - \frac{1+i}{2+i} r_1, r_3 \to r_3 - \frac{1+2i}{2+i} r_1$, giving

$$\begin{pmatrix} 2+i & -1+2i & 2 \\ 0 & -1+i-\frac{(1+i)(-1+2i)}{2+i} & 1-\frac{2+2i}{2+i} \\ 0 & -2+i-\frac{(1+2i)(-1+2i)}{2+i} & 1+i-\frac{2+4i}{2+i} \end{pmatrix}$$
$$= \begin{pmatrix} 2+i & -1+2i & 2 \\ 0 & 0 & -\frac{1}{5}(-1+2i) \\ 0 & 0 & -\frac{1}{5}(3+i) \end{pmatrix}$$

and then do $r_3 \to r_3 - \frac{3+i}{-1+2i} r_2$, giving

$$\begin{pmatrix} 2+i & -1+2i & 2 \\ 0 & 0 & -\frac{1}{5}(-1+2i) \\ 0 & 0 & 0 \end{pmatrix},$$

which is an equivalent r.e.f. matrix.

(3) First find the inverse of the matrix given.

$$
\begin{array}{r c}
 & \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \\
\begin{array}{r} r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - r_1 \end{array} & \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 4 & -2 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \\
r_3 \to r_3 - \frac{1}{3}r_2 & \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 4 & -2 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right) \\
r_3 \to \frac{3}{2}r_3 & \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 4 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{array}\right) \\
r_2 \to r_2 - 4r_3 & \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{array}\right) \\
r_2 \to \frac{1}{3}r_2 & \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{array}\right) \\
r_1 \to r_1 + r_3 & \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{array}\right)
\end{array}
$$

(4)

$$A\mathbf{x} = \mathbf{b} \Rightarrow LU\mathbf{x} = \mathbf{b}$$

Set $\mathbf{y} = U\mathbf{x}$ then solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

(5) **(a)** Let $U = (u_{ij})$ and $V = (v_{ij})$ be upper triangular matrices i.e. $u_{ij} = v_{ij} = 0$ for $1 \leq j < i \leq n$. Now $(UV)_{ij} = \Sigma_{k=1}^{n} u_{ik}v_{kj}$, and in this sum $u_{ik} = 0$ for $1 \leq k < i$ and $v_{kj} = 0$ for $j < k \leq n$. So when $i > j$ all summands are 0 and thus $(UV)_{ij} = 0$. Hence UV is upper triangular.

For the inverse of U, the jth column of U^{-1} is the solution $\mathbf{x}$ of $U\mathbf{x} = \mathbf{e}_j$. Since $(\mathbf{e}_j)_i = 0$ for i with $j < i \leq n$, solution by back substitution gives $x_n = x_{n-1} = \cdots = x_{j+1} = 0$. Since $x_i = (U^{-1})_{ij}$ we have $(U^{-1})_{ij} = 0$ if $j < i \leq n$, so U^{-1} is upper triangular.

(b) If L and M are lower triangular, L^T and M^T are upper triangular, so $M^T L^T$ is upper triangular by (a), and hence $LM = (M^T L^T)^T$ is lower triangular. Also $(L^T)^{-1}$ is upper triangular, so $L^{-1} = ((L^T)^{-1})^T$ is lower triangular.

(c) Put $L = (l_{ij})$ so $l_{ij} = 0$ if $1 \le i < j \le n$ and $l_{ii} = 1$. The jth column of L^{-1} is the solution $\mathbf{x}$ of $L\mathbf{x} = \mathbf{e}_j$. The first j equations of this system are $x_1 = 0, l_{21}x_1 + x_2 = 0, l_{31}x_1 + l_{32}x_2 + x_3 = 0, \ldots, l_{j1}x_1 + l_{j2}x_2 + \cdots + l_{jj-1}x_{j-1} + x_j = 1$. So $x_1 = x_2 = \cdots = x_{j-1} = 0, x_j = 1$. Hence $(L^{-1})_{jj} = x_j = 1$.

(6) **(a)** In this case, we can use Gaussian elimination with no row interchanges. As in lectures, we record the steps in compact form: eg, when $R_2 \to R_2 - cR_1$ makes the $(2,1)$ entry 0, place c there, but it is really zero!

$$\begin{pmatrix} 3 & 1 & 0 & -5 \\ -6 & -1 & -1 & 10 \\ 3 & 3 & 2a-2 & a-5 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{matrix} \\ R_2 \to R_2 - (-2)R_1 \\ R_3 \to R_3 - 1R_1 \\ \\ \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 3 & 1 & 0 & -5 \\ -2 & 1 & -1 & 0 \\ 1 & 2 & 2a-2 & a \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - (-1)R_2 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 3 & 1 & 0 & -5 \\ -2 & 1 & -1 & 0 \\ 1 & 2 & 2a & a \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

The entries of L below the main diagonal are then all the entries with a circle around them. Hence $A = LU$ where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 1 & 0 & -5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2a & a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Since $\det(A) = \det(L)\det(U)$, with $\det(L) = 1$ and $\det(U) = 3 \times 1 \times 2a \times 0 = 0$ (ie. the product of the diagonal entries), we have

$$\det(A) = 0.$$

(c) Set $\mathbf{y} = U\mathbf{x}$ and first solve $L\mathbf{y} = \mathbf{b}$, that is,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{aligned} \text{row 1} \quad &\Rightarrow \quad y_1 = 1 \\ \text{row 2} \quad &\Rightarrow \quad -2y_1 + y_2 = -2 \;\Rightarrow\; y_2 = 0 \\ \text{row 3} \quad &\Rightarrow \quad y_1 + 2y_2 + y_3 = 1 \;\Rightarrow\; y_3 = 0 \\ \text{row 4} \quad &\Rightarrow \quad -2y_2 + y_4 = 0 \;\Rightarrow\; y_4 = 0. \end{aligned}$$

Now solve $U\mathbf{x} = \mathbf{y}$, that is

$$\begin{pmatrix} 3 & 1 & 0 & -5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2a & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{row 3} \quad &\Rightarrow \quad 2ax_3 + ax_4 = 0 \;\Rightarrow\; x_4 = -2x_3 \;(a \neq 0) \\ \text{row 2} \quad &\Rightarrow \quad x_2 - x_3 = 0 \;\Rightarrow\; x_2 = x_3 \\ \text{row 1} \quad &\Rightarrow \quad 3x_1 + x_2 - 5x_4 = 1 \;\Rightarrow\; x_1 = \frac{1}{3}(1 - 11x_3). \end{aligned}$$

Set $x_3 = t$, then the solution is

$$\mathbf{x} = \begin{pmatrix} \frac{1}{3}(1 - 11t) \\ t \\ t \\ -2t \end{pmatrix}, \quad \text{for any } t.$$

(d) Using the same method as in part (c), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} y_1 &= 2 \\ y_2 &= -1 \\ y_3 &= 0 \\ y_4 &= 0 \end{aligned}$$

Now solve $U\mathbf{x} = \mathbf{y}$. That is,

$$\begin{pmatrix} 3 & 1 & 0 & -5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2a & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} x_4 &= -2x_3 \\ x_2 &= x_3 - 1 \\ x_1 &= \tfrac{1}{3}(3 - 11x_3) \end{aligned}$$

Set $x_3 = t$, so that the solution is

$$\mathbf{x} = \begin{pmatrix} \frac{1}{3}(3-11t) \\ t-1 \\ t \\ -2t \end{pmatrix}, \quad \text{for any } t.$$

(7) Try doing Gaussian elimination, using the same notation as in lectures and the previous question:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 8 & 12 & 17 \\ 3 & 6 & 12 & 14 \\ 2 & 9 & 11 & 12 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 0 & 0 & 1 \\ 3 & 0 & 3 & 2 \\ 2 & 5 & 5 & 4 \end{pmatrix}$$

In order to continue the Gaussian reduction, we need to swap rows 2 and 4. So after $R_2 \leftrightarrow R_4$,

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 5 & 4 \\ 3 & 0 & 3 & 2 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

which is then in r.e.f. In other words, we have

$$PA = LU,$$

where $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Also, the decomposition is $A = PLU$, since $P^2 = I$.

This is the same L and U we would obtain if we had swapped rows 2 and 4 initially. However, it was difficult to predict that we needed this operation at the beginning.

$$\det(A) = -\det(U) = -1 \times 5 \times 3 \times 1 = -15,$$

where we need the minus sign since we have used an odd number of row swaps (in this case only one row swap).

(8) Part (a) In this case, we can use Gaussian elimination with no row interchanges. As in lectures, we record the steps in compact form: eg, when $R_2 \to R_2 - cR_1$ makes the $(2,1)$ entry 0, place c there, but it is really zero!

$$\begin{pmatrix} -1 & -3 & -4 \\ 3 & 10 & -10 \\ -2 & -4 & a \end{pmatrix} \begin{matrix} R_2 \to R_2 - (-3)R_1 \\ R_3 \to R_3 - (+2)R_1 \end{matrix}$$

$$\begin{pmatrix} -1 & -3 & -4 \\ -3 & 1 & -22 \\ +2 & 2 & a+8 \end{pmatrix} \quad R_3 \to R_3 - (+2)R_2$$

$$\begin{pmatrix} -1 & -3 & -4 \\ -3 & 1 & -22 \\ +2 & +2 & a+52 \end{pmatrix}$$

The entries of L below the main diagonal are then all the entries with a circle around them. Hence $A = LU$ where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \quad U = \begin{pmatrix} -1 & -3 & -4 \\ 0 & 1 & -22 \\ 0 & 0 & a+52 \end{pmatrix}$$

Part (b)

$\det(A) = \det(L)\det(U) = \det(U) = -a - 52$

so $\det(A) = 0$ when $a = -52$

Part (c)

To solve $LUx = b$ set $Ux = y$ and solve $Ly = b$.

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 9 \end{pmatrix},$$

$$\begin{array}{lll} \text{row 1} & \Rightarrow & y_1 = -6 \\ \text{row 2} & \Rightarrow & -3y_1 + y_2 = -3 \ \Rightarrow \ y_2 = -21 \\ \text{row 3} & \Rightarrow & 2y_1 + 2y_2 + y_3 = 9 \ \Rightarrow \ y_3 = 63 \end{array}$$

Then solve $Ux = y$

$$\begin{pmatrix} -1 & -3 & -4 \\ 0 & 1 & -22 \\ 0 & 0 & 63 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6 \\ -21 \\ 63 \end{pmatrix},$$

$$\begin{aligned} &\text{row 3} \quad \Rightarrow \quad x_3 = 1 \\ &\text{row 2} \quad \Rightarrow \quad x_2 - 22x_3 = -21 \Rightarrow \ x_2 = 1 \\ &\text{row 1} \quad \Rightarrow \quad -x_1 - 3x_2 - 4x_3 = -6 \Rightarrow \ x_1 = -1 \end{aligned}$$

So

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

(9) As in the example above, we record the steps in compact form.

$$\begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{pmatrix} \quad R_3 \to R_3 - (+1)R_1$$

$$\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ +1 & 0 & 1 \end{pmatrix} \quad \text{(R.E.F)}$$

This gives

$U = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Where $PA = LU$

NOTE :

We have $U = EPA$ where

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \ P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

so $PA = E^{-1}U$ where

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = L.$$