DEPARTMENT OF MATHEMATICS

MATH2000
Green's theorem, introduction to flux (solutions).

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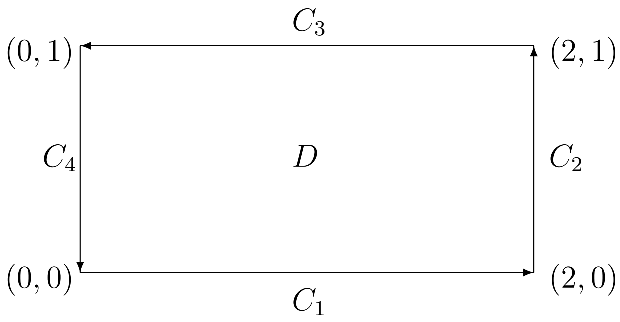
(1) Work done $= \int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (y - xy)\mathbf{i} + x^2\boldsymbol{j}$ and C is the rectangle in the x-y plane:



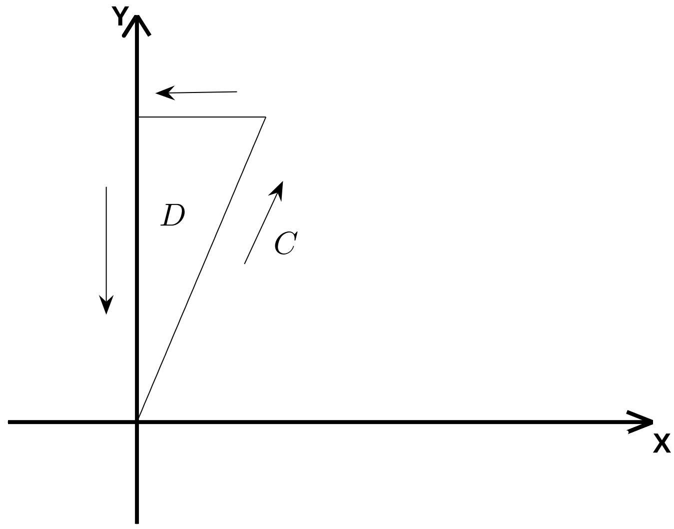
Instead of evaluating $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$, just use Green's theorem in the plane:

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy$$

with $D = \{(x, y) \mid 0 \leq 2, 0 \leq y \leq 1\}$.
Now $\frac{\partial F_2}{\partial x} = 2x$ and $\frac{\partial F_1}{\partial y} = 1 - x$, so work done

$$\begin{aligned}
&= \int_0^1 \int_0^2 (3x - 1) dxdy \quad \text{(Green's Theorem)} \\
&= \int_0^1 \left[\frac{3}{2}x^2 - x \right]_0^2 dy \\
&= \int_0^1 4dy \\
&= 4.
\end{aligned}$$

(2) The curve C traverses the outside of the region D in the diagram below, in a counter-clockwise direction:



The region D can be expressed as

$$D = \{(x, y) \mid 0 \leq x \leq 1, 3x \leq y \leq 3\}.$$

With $F_1 = x^2y^2$ and $F_2 = 4xy^3$, we have

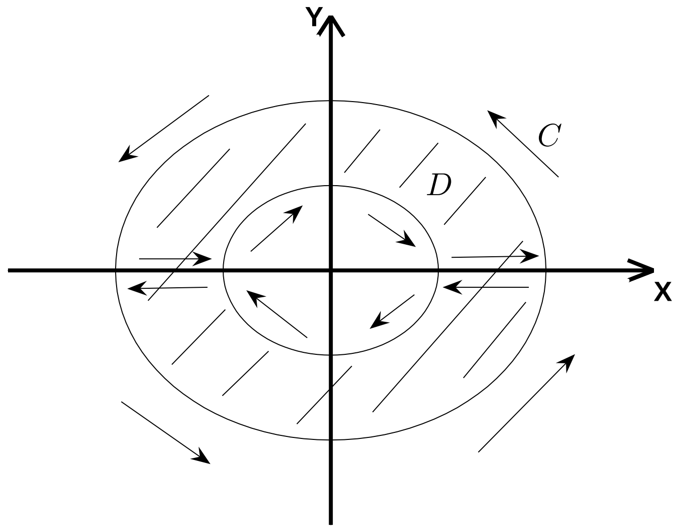
$$\frac{\partial F_1}{\partial y} = 2x^2y, \ \frac{\partial F_2}{\partial x} = 4y^3.$$

By Green's theorem, we have

$$\begin{aligned}
\int_C x^2y^2 \ dx + 4xy^3 \ dy &= \int_0^1 \int_{3x}^3 (4y^3 - 2x^2y)dy \ dx \\
&= \int_0^1 \left[y^4 - x^2y^2\right]_{3x}^3 dx \\
&= \int_0^1 (81 - 9x^2 - 72x^4)dx \\
&= \left[81x - 3x^3 - \frac{72}{5}x^5\right]_0^1 = \frac{318}{5}.
\end{aligned}$$

(3) The diagram below describes the curve C, which actually consists of two curves, bounding a circular region D with a hole. To be positively oriented, the curve must always have the region D to the left. In other words, C traverses in a counterclockwise direction around the circle of radius 2, and a clockwise direction over the circle of radius 1.

To be able to use Green's theorem correctly, we divide D into two subregions (above and below the x axis) by extending the boundaries along the positive and negative x axes as indicated by the arrows. The line integrals along these parts of the curve will cancel, because there will be two line itegrals over the same curves but in opposite directions.



The region D can be expressed nicely using polar coordinates as

$$D = \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le 2\pi\}.$$

We also have $F_1 = xe^{-2x}$, $F_2 = x^4 + 2x^2y^2$ so that

$$\frac{\partial F_1}{\partial y} = 0, \quad \frac{\partial F_2}{\partial x} = 4x^3 + 4xy^2.$$

We then have

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 4x^3 + 4xy^2 = 4x(x^2 + y^2).$$

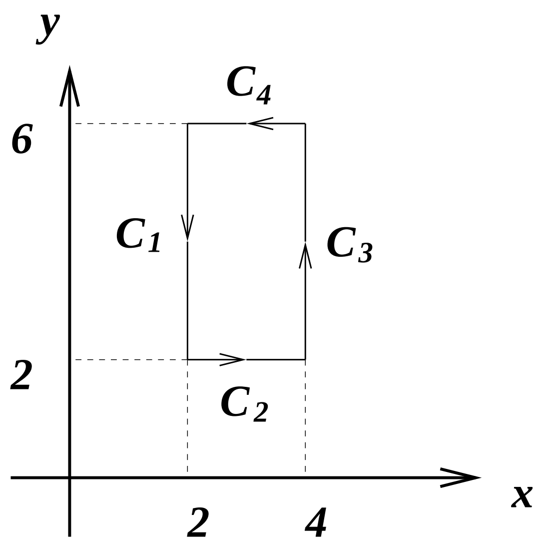
Using polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$), we can express this as

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 4r\cos\theta \times r^2.$$

By Green's theorem (and using polar coordinates) we have

$$\int_C xe^{-2x}\, dx + (x^4 + 2x^2y^2)\, dy \;=\; \int_1^2 \int_0^{2\pi} 4r^3 \cos\theta \; r \; d\theta \; dr = 0, \text{ since } \int_0^{2\pi} \cos\theta \; d\theta = 0.$$

(4) The diagram below shows the path of integration: we shall choose to traverse the boundary of the rectangle in a counterclockwise direction.



To calculate the net outward flux, we shall calculate the outwardly directed flux across each of the four curves which make up the boundary of the rectangle. That is, we shall calculate

$$\text{net flux } = \int_{C_1} \boldsymbol{v}\cdot\boldsymbol{n}_1\, dS + \int_{C_2} \boldsymbol{v}\cdot\boldsymbol{n}_2\, dS + \int_{C_3} \boldsymbol{v}\cdot\boldsymbol{n}_3\, dS + \int_{C_4} \boldsymbol{v}\cdot\boldsymbol{n}_4\, dS,$$

where $\boldsymbol{n}_i$ is an outwardly directed unit normal vector to the line C_i.

For C_1:

First parametrise the curve as

$$\boldsymbol{r}(t) = 2\boldsymbol{i} + (6-t)\boldsymbol{j}\ \ (0 \le t \le 4)\ \Rightarrow\ \boldsymbol{r}'(t) = -\boldsymbol{j},$$

so in this case $\boldsymbol{r}'(t)$ is a unit tangent vector. The outwardly pointing unit normal vector is $\boldsymbol{n}_1 = -\boldsymbol{i}\ (\ = (-\boldsymbol{j})\times\boldsymbol{k})$. With this parametrisation,

$$\boldsymbol{v} = (t-6)\boldsymbol{i} + 2\boldsymbol{j}\ \Rightarrow\ \boldsymbol{v}\cdot\boldsymbol{n}_1 = 6-t.$$

As always, within the integral the infinitesimal element of arc can be expressed in terms of the parametrisation as $dS = |\boldsymbol{r}'(t)|dt = dt$ in this case. The outwardly directed flux across C_1 is then

$$\int_0^4 (6-t)\ dt = 16.$$

For C_2:

First parametrise the curve as

$$\boldsymbol{r}(t) = t\boldsymbol{i} + 2\boldsymbol{j} \ (2 \le t \le 4) \ \Rightarrow \ \boldsymbol{r}'(t) = \boldsymbol{i},$$

so in this case $\boldsymbol{r}'(t)$ is a unit tangent vector. The outwardly pointing unit normal vector is $\boldsymbol{n}_2 = -\boldsymbol{j}$ ($= \boldsymbol{i} \times \boldsymbol{k}$). With this parametrisation,

$$\boldsymbol{v} = -2\boldsymbol{i} + t\boldsymbol{j} \ \Rightarrow \ \boldsymbol{v} \cdot \boldsymbol{n}_2 = -t.$$

As always, within the integral the infinitesimal element of arc can be expressed in terms of the parametrisation as $dS = |\boldsymbol{r}'(t)|dt = dt$ in this case. The outwardly directed flux across C_2 is then

$$\int_2^4 (-t)\ dt = -6.$$

For C_3:

First parametrise the curve as

$$\boldsymbol{r}(t) = 4\boldsymbol{i} + t\boldsymbol{j} \ (2 \le t \le 6) \ \Rightarrow \ \boldsymbol{r}'(t) = \boldsymbol{j},$$

so in this case $\boldsymbol{r}'(t)$ is a unit tangent vector. The outwardly pointing unit normal vector is $\boldsymbol{n}_3 = \boldsymbol{i}$ ($= \boldsymbol{j} \times \boldsymbol{k}$). With this parametrisation,

$$\boldsymbol{v} = -t\boldsymbol{i} + 4\boldsymbol{j} \ \Rightarrow \ \boldsymbol{v} \cdot \boldsymbol{n}_3 = -t.$$

As always, within the integral the infinitesimal element of arc can be expressed in terms of the parametrisation as $dS = |\boldsymbol{r}'(t)|dt = dt$ in this case. The outwardly directed flux across C_3 is then

$$\int_2^6 (-t)\ dt = -16.$$

For C_4:

First parametrise the curve as

$$\boldsymbol{r}(t) = (4-t)\boldsymbol{i} + 6\boldsymbol{j} \ (0 \le t \le 2) \ \Rightarrow \ \boldsymbol{r}'(t) = -\boldsymbol{i},$$

so in this case $\boldsymbol{r}'(t)$ is a unit tangent vector. The outwardly pointing unit normal vector is $\boldsymbol{n}_4 = \boldsymbol{j}$ ($= (-\boldsymbol{i}) \times \boldsymbol{k}$). With this parametrisation,

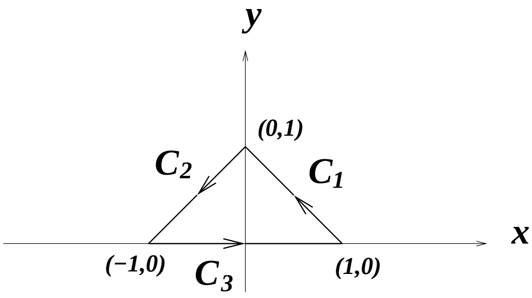
$$\boldsymbol{v} = -6\boldsymbol{i} + (4-t)\boldsymbol{j} \ \Rightarrow \ \boldsymbol{v} \cdot \boldsymbol{n}_4 = 4 - t.$$

As always, within the integral the infinitesimal element of arc can be expressed in terms of the parametrisation as $dS = |\boldsymbol{r}'(t)|dt = dt$ in this case. The outwardly directed flux across C_4 is then

$$\int_0^2 (4-t)\ dt = 6.$$

Therefore the net outward flux is $16 + (-6) + (-16) + 6 = 0$.

(5) The diagram below shows the path of integration: we shall choose to traverse the boundary of the rectangle in a counterclockwise direction.



To calculate the net outward flux, we shall calculate the outwardly directed flux across each of the three curves which make up the boundary of the triangle. That is, we shall calculate

$$\text{net flux } = \int_{C_1} \boldsymbol{v} \cdot \boldsymbol{n}_1 \, dS + \int_{C_2} \boldsymbol{v} \cdot \boldsymbol{n}_2 \, dS + \int_{C_3} \boldsymbol{v} \cdot \boldsymbol{n}_3 \, dS,$$

where $\boldsymbol{n}_i$ is an outwardly directed unit normal vector to the line C_i.

For C_1:

Since this is part of the straight line with equation $x + y = 1$, we can first parametrise the curve as

$$\boldsymbol{r}(t) = (1-t)\boldsymbol{i} + t\boldsymbol{j} \ (0 \le t \le 1) \ \Rightarrow \ \boldsymbol{r}'(t) = -\boldsymbol{i} + \boldsymbol{j}.$$

To work out $\boldsymbol{n}_1$, we need a unit tangent vector $\boldsymbol{T}_1$ so we can then take $\boldsymbol{n}_1 = \boldsymbol{T}_1 \times \boldsymbol{k}$. Since $\boldsymbol{r}'(t)$ is a tangent vector, we take

$$\boldsymbol{T}_1 = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{-\boldsymbol{i} + \boldsymbol{j}}{\sqrt{2}}.$$

The outwardly pointing unit normal vector is then

$$\boldsymbol{n}_1 = \boldsymbol{T}_1 \times \boldsymbol{k} = \frac{1}{\sqrt{2}} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{\boldsymbol{i} + \boldsymbol{j}}{\sqrt{2}}.$$

With this parametrisation,

$$\boldsymbol{v} = \boldsymbol{i} - (1 - 2t + 2t^2)\boldsymbol{j} \ \Rightarrow \ \boldsymbol{v} \cdot \boldsymbol{n}_1 = \frac{2t - 2t^2}{\sqrt{2}}.$$

As always, within the integral the infinitesimal element of arc can be expressed in terms of the parametrisation as $dS = |\boldsymbol{r}'(t)|dt = \sqrt{2}\ dt$ in this case. The outwardly directed flux across C_1 is then

$$\int_0^1 (2t - 2t^2) dt = \frac{1}{3}.$$

For C_2:

Since this is part of the straight line with equation $y = 1 + x$, we can first parametrise the curve as

$$\boldsymbol{r}(t) = -t\boldsymbol{i} + (1-t)\boldsymbol{j} \ (0 \le t \le 1) \ \Rightarrow \ \boldsymbol{r}'(t) = -\boldsymbol{i} - \boldsymbol{j}.$$

To work out $\boldsymbol{n}_2$, we need a unit tangent vector $\boldsymbol{T}_2$ so we can then take $\boldsymbol{n}_2 = \boldsymbol{T}_2 \times \boldsymbol{k}$. Since $\boldsymbol{r}'(t)$ is a tangent vector, we take

$$\boldsymbol{T}_2 = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{-\boldsymbol{i} - \boldsymbol{j}}{\sqrt{2}}.$$

The outwardly pointing unit normal vector is then

$$\boldsymbol{n}_2 = \boldsymbol{T}_2 \times \boldsymbol{k} = \frac{1}{\sqrt{2}} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{-\boldsymbol{i} + \boldsymbol{j}}{\sqrt{2}}.$$

With this parametrisation,

$$\boldsymbol{v} = (1-2t)\boldsymbol{i} - (1 - 2t + 2t^2)\boldsymbol{j} \ \Rightarrow \ \boldsymbol{v} \cdot \boldsymbol{n}_2 = \frac{-2 + 4t - 2t^2}{\sqrt{2}}.$$

As always, within the integral the infinitesimal element of arc can be expressed in terms of the parametrisation as $dS = |\boldsymbol{r}'(t)|dt = \sqrt{2}\ dt$ in this case. The outwardly directed flux across C_2 is then

$$\int_0^1 (-2 + 4t - 2t^2)dt = -\frac{2}{3}.$$

For C_3:

Since this is part of the straight line with equation $y = 0$, we can first parametrise the curve as

$$\boldsymbol{r}(t) = (t-1)\boldsymbol{i} \ (0 \le t \le 2) \ \Rightarrow \ \boldsymbol{r}'(t) = \boldsymbol{i},$$

so in this case $\boldsymbol{r}'(t)$ is a unit tangent vector. The outwardly pointing unit normal vector is $\boldsymbol{n}_3 = -\boldsymbol{j}\ (= \boldsymbol{i} \times \boldsymbol{k})$.

With this parametrisation,

$$\boldsymbol{v} = (t-1)\boldsymbol{i} - (t^2 - 2t + 1)\boldsymbol{j} \ \Rightarrow \ \boldsymbol{v} \cdot \boldsymbol{n}_2 = t^2 - 2t + 1.$$

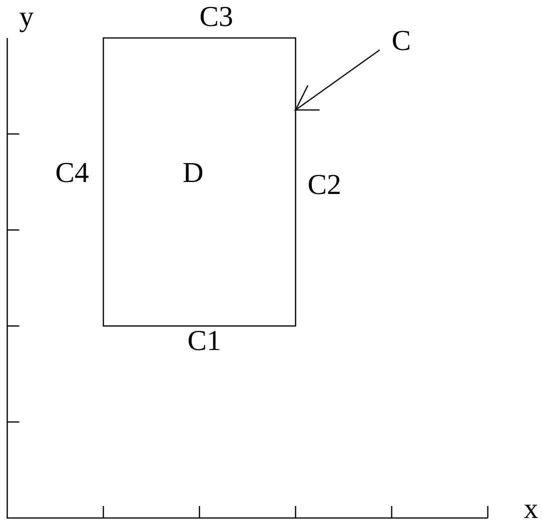
As always, within the integral the infinitesimal element of arc can be expressed in terms of the parametrisation as $dS = |\boldsymbol{r}'(t)|dt = dt$ in this case. The outwardly directed flux across C_3 is then

$$\int_0^2 (t^2 - 2t + 1)dt = \frac{2}{3}.$$

Therefore the net outward flux is $\displaystyle \frac{1}{3} + \left(-\frac{2}{3}\right) + \frac{2}{3} = \frac{1}{3}.$

(6)

Referring to the figure



$$Flux = \oint \alpha x\hat{i}\cdot\hat{n}ds = \int_{C_1}\alpha x\hat{i}\cdot\hat{n}_1 ds + \int_{C_2}\alpha x\hat{i}\cdot\hat{n}_2 ds + \int_{C_3}\alpha x\hat{i}\cdot\hat{n}_3 ds + \int_{C_4}\alpha x\hat{i}\cdot\hat{n}_4 ds$$

Where $\hat{n}_i$ is the outward unit normal to the line C_i.

$$Flux = \int_1^3 \alpha x\hat{i}\cdot(-\hat{j})dx + \int_2^5 3\alpha\hat{i}\cdot(\hat{i})dy + \int_1^3 \alpha x\hat{i}\cdot(\hat{j})dx + \int_2^5 1\alpha\hat{i}\cdot(-\hat{i})dy$$

$$= 0 + 3\alpha\int_2^5 dy + 0 - \alpha\int_2^5 dy$$

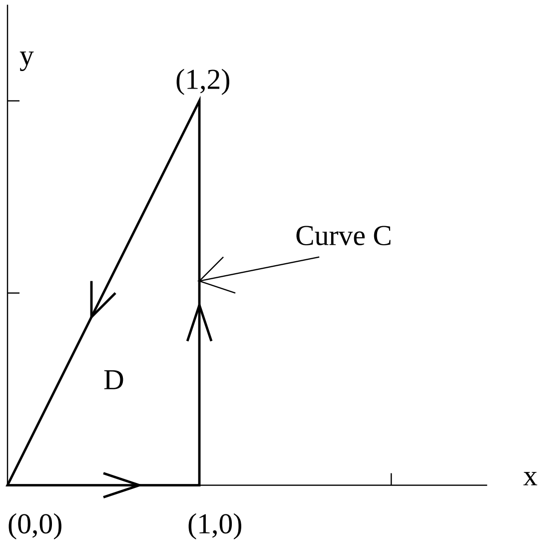
$$= 9\alpha - 3\alpha = 6\alpha$$

The dimensions are in m^2/s

NOTE : This problem could also be solved using the flux form of Greens theorem.

$$\oint ax\hat{i}\cdot\hat{n}dl = \int\int_D \nabla\cdot\left(\alpha x\hat{i}\right)dA = \int_1^3\int_2^5 \alpha dydx = 6\alpha$$

(7)



$$\int_C xydx + x^2y^3dy = \int_C \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{F} = (F_1, F_2) = (xy, x^2y^3)$ and C is the curve indicated in the diagram. Green's theorem in the plane is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

where D is the area enclosed by C. In this case

$$D = \{(x,y)|0 \le x \le 1, 0 \le y \le 2x\}$$

$$F_1 = xy \implies \frac{\partial F_1}{\partial y} = x$$

$$F_2 = x^2y^3 \implies \frac{\partial F_2}{\partial x} = 2xy^3$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2xy^3 - x$$

$$\oint xydx + x^2y^3dy = \int (2xy^3 - x)dA = \int_0^1 \int_0^{2x} (2xy^3 - x)dydx$$

$$= \int_0^1 \Big|_0^{2x} \left[\frac{1}{2}xy^4 - xy\right] dx$$

$$= \int_0^1 8x^5 - 2x^2dx = \Big|_0^1 \left[\frac{8}{6}x^6 - \frac{2}{3}x^3\right] = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$