DEPARTMENT OF MATHEMATICS

MATH2000 Hyperbolic functions solutions

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(1) For $\int \frac{xdx}{\sqrt{1+x^4}}$, set $u = x^2 \Rightarrow du = 2xdx$

$$\Rightarrow \int \frac{x\ dx}{\sqrt{1+x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1+u^2}}$$

Now set $u = \sinh t \Rightarrow du = \cosh t dt$. Now the integral is:

$$\frac{1}{2}\int \frac{\cosh t dt}{\sqrt{1+\sinh^2 t}}$$

$$= \frac{1}{2}\int dt$$

$$= \frac{t}{2} + c = \frac{\text{arsinh} u}{2} + c$$

Hence

$$\int \frac{xdx}{\sqrt{1+x^4}} = \frac{\text{arsinh} x^2}{2} + c.$$

(2) Let $y = \text{arcosh} x \Rightarrow x = \cosh y$ (for $y \geq 0, x \geq 1$)

$$\Rightarrow x = \frac{e^y + e^{-y}}{2}$$

$$\Rightarrow e^y - 2x + e^{-y} = 0$$

$$\Rightarrow e^{2y} - 2xe^y + 1 = 0$$

Using the quadratic formula:

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2-1}$$

But since $y \geq 0, e^y \geq 1\ \forall y$, and $x - \sqrt{x^2-1} < 1$ for $x > 1$, we have

$$e^y = x + \sqrt{x^2-1}$$

$\Rightarrow y = \ln\left(x + \sqrt{x^2-1}\right)$ as required.

(3) Using the hint:

$$\int \text{arsinh}x dx = \int 1.\text{arsinh}x dx$$

Setting $u' = 1, v = \text{arsinh}x$, we get $u = x, v' = \frac{1}{\sqrt{1+x^2}}$

$$\Rightarrow \int \text{arsinh}x dx = x.\text{arsinh}x - \int \frac{xdx}{\sqrt{1+x^2}}$$

Setting $w = 1 + x^2 \Rightarrow dw = 2x$.

$$\Rightarrow \int \text{arsinh}x dx = x.\text{arsinh}x - \frac{1}{2}\int \frac{dw}{\sqrt{w}}$$

$$= x.\text{arsinh}x - \sqrt{1+x^2} + c$$

(4) For $\frac{d^2y}{dx^2} = 1 - \left(\frac{dy}{dx}\right)^2$, set $u = \frac{dy}{dx}$

$$\Rightarrow \frac{du}{dx} = 1 - u^2$$

$$\Rightarrow \int \frac{du}{1-u^2} = \int dx$$

$$\Rightarrow \text{artanh } u = x + c \quad (\text{note } |u| < 1)$$

$$\Rightarrow u = \tanh(x+c)$$

$$\Rightarrow \frac{dy}{dx} = \tanh(x+c)$$

$$\Rightarrow y = \int \frac{\sinh(x+c)\ dx}{\cosh(x+c)}$$

$$= \ln|\cosh(x+c)| + k.$$

Note that having $u = \pm 1$ gives a solution $y(x) = \pm x + c$. However, these don't satisfy the condition $\left|\frac{dy}{dx}\right| < 1$.

(5) For $\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^2 + 4 = 0$, set $u = \frac{dy}{dx}$. Then:

$$(u')^2 = u^2 - 4$$

$$u' = \pm\sqrt{u^2 - 4}$$

$$\int \frac{du}{\sqrt{u^2-4}} = \pm \int dx$$

Setting $u = 2\cosh t$ (note $|u| > 2$ is given), $u^2 - 4 = 4\cosh^2 t - 4 = 4\sinh^2 t$

$$du = 2\sinh t dt$$

$$\Rightarrow \int \frac{du}{\sqrt{u^2-4}} = \int dt = \text{ R.H.S. } = \pm \int dx$$

$$\Rightarrow t = \pm x + c$$

$$\Rightarrow \text{ arcosh}\left(\frac{u}{2}\right) = \pm x + c$$

$$\Rightarrow u = 2\cosh(\pm x + c).$$

For u to satisfy the given condition $|u| > 2$ for $x \neq 0$, we must have $c = 0$. Note also that cosh is an even function, so that $\cosh(-x) = \cosh x$. Therefore without loss of generality, we have

$$u = 2\cosh(x)$$

$$\Rightarrow y = 2\int \cosh(x) dx$$

$$\Rightarrow y = 2\sinh(x) + k.$$

Note that $y(x) = \pm 2x + c$ would also satisfy the equation, but not the condition $\left|\frac{dy}{dx}\right| > 2$.

(6) (a) Recall the Taylor series expansion for e^x:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \dots$$

$$\Rightarrow e^{ix} = 1 + ix + \frac{1}{2}i^2x^2 + \frac{1}{3!}i^3x^3 + \frac{1}{4!}i^4x^4 + \frac{1}{5!}i^5x^5 + \frac{1}{6!}i^6x^6 + \dots$$

$$\text{Now } i^2 = -1, \quad i^3 = i \times i^2 = i(-1) = -i, \quad i^4 = i \times i^3 = i(-i) = -i^2 = 1,$$

$$i^5 = i \times i^4 = i(1) = i, \quad i^6 = i \times i^5 = i(i) = i^2 = -1 \text{ etc.}$$

$$\Rightarrow e^{ix} = \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right) + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots\right)$$

Since the terms in brackets are the Taylor series of $\cos x$ and $\sin x$ respectively, we have

$$e^{ix} = \cos x + i\sin x.$$

Recall the Taylor series for e^x, $\sin x$ and $\cos x$ converge for any x.

(b) By definition

$$\text{RHS } = 2 \times \frac{e^x - e^{-x}}{2} \times \frac{e^x + e^{-x}}{2} = \frac{(e^x)^2 - (e^{-x})^2)}{2} = \frac{e^{2x} - e^{-2x}}{2} = \text{ LHS}$$

(c) Since $\sinh(2ix) = i\sin(2x)$ we have

$$i\sin(2x) = 2\sinh(ix)\cosh(ix) = 2i\sin x\cos x$$

$$\Rightarrow \sin(2x) = 2\sin x\cos x.$$