DEPARTMENT OF MATHEMATICS

Revision solutions

(1) For this example we use partial fractions. We can write

$$\begin{aligned}
\frac{1}{(1-x)(2+x)} &= \frac{A}{1-x}+\frac{B}{2+x}\\
\Rightarrow\ 1 &= A(2+x)+B(1-x)\\
\Rightarrow\ 2A+B=1 &\ \ \&\ \ A-B=0\\
\Rightarrow\ A=B &= \frac{1}{3}\\
\Rightarrow\ \int\frac{dx}{(1-x)(2+x)} &= \frac{1}{3}\int\frac{dx}{1-x}+\frac{1}{3}\int\frac{dx}{2+x}\\
&= -\frac{1}{3}\ln|1-x|+\frac{1}{3}\ln|2+x|+c.
\end{aligned}$$

(2) Making the substitution $u=x^3$ means within the integral we can also make the substitution $du=3x^2dx$. Therefore

$$\begin{aligned}
\int x^2e^{x^3}dx &= \frac{1}{3}\int e^u du\\
&= \frac{1}{3}e^u+c\\
&= \frac{1}{3}e^{x^3}+c.
\end{aligned}$$

(3) Based on the trigonometric identity $\tan^2\theta+1=\sec^2\theta$, we make the substitution $x=\tan\theta$ so that within the integral we also have $dx=\sec^2\theta d\theta$. Therefore

$$\begin{aligned}
\int\frac{dx}{x^2+1} &= \int\frac{\sec^2 d\theta}{\tan^2\theta+1}\\
&= \int\frac{\sec^2\theta}{\sec^2\theta}d\theta\\
&= \int d\theta\\
&= \theta+c\\
&= \arctan x+c.
\end{aligned}$$

(4) Once we factorise $4-x^2=(2+x)(2-x)$ it should be clear that we can use partial fractions. We have

$$\begin{aligned}
\frac{1}{(2+x)(2-x)} &= \frac{A}{2+x}+\frac{B}{2-x}\\
\Rightarrow\ 1 &= A(2-x)+B(2+x)\\
\Rightarrow\ 2A+2B=1 &\ \ \&\ \ B-A=0\\
\Rightarrow\ A=B &= \frac{1}{4}\\
\Rightarrow\ \int\frac{dx}{(2+x)(2-x)} &= \frac{1}{4}\int\frac{dx}{2+x}+\frac{1}{4}\int\frac{dx}{2-x}\\
&= \frac{1}{4}\ln|2+x|-\frac{1}{4}\ln|2-x|+c.
\end{aligned}$$

(5) This example is similar to the previous one, with $x^2-9=(x+3)(x-3)$, except we now have

$$\frac{1}{(x+3)(x-3)} = \frac{\frac{1}{6}}{x-3} - \frac{\frac{1}{6}}{x+3},$$

leading to

$$\int \frac{dx}{x^2-9} = \frac{1}{6}\int \frac{dx}{x-3} - \frac{1}{6}\int \frac{dx}{x+3} = \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + c.$$

(6) Note that

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

which suggests that the substitution $u = \cos x$ will be useful. In that case we have $du = -\sin x dx$ within the integral so that

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
&= -\int \frac{du}{u} \\
&= -\ln|u| + c \\
&= -\ln|\cos x| + c \\
&= \ln|(\cos x)^{-1}| + c \\
&= \ln|\sec x| + c.
\end{aligned}$$

(7) This integral involves a little trick. Recall integration by parts:

$$\int u' \cdot v dx = u \cdot v - \int u \cdot v' dx,$$

where u and v are both functions of x. For this example we set $u' = 1$ and $v = \ln x$. Therefore

$$\begin{aligned}
\int \ln x dx &= \int 1 \cdot \ln x dx \\
&= x\ln x - \int x \cdot \left(\frac{1}{x}\right) dx \\
&= x \ln x - x + c.
\end{aligned}$$

(8) It is worth remembering the identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$. We then have

$$\begin{aligned}
\int \cos^2 x dx &= \frac{1}{2}\int (1+\cos 2x) dx \\
&= \frac{x}{2} + \frac{1}{4}\sin 2x + c.
\end{aligned}$$

(9) Once again we use integration by parts (see (7) above), but this time we need to use it more than once. We start with $u' = e^{-x}$ and $v = x^2$:

$$\begin{aligned}
\int x^2 e^{-x} dx &= -x^2 e^{-x} - \int (2x)(-e^{-x}) dx \\
&= -x^2 e^{-x} + 2\int x e^{-x} dx.
\end{aligned}$$

Now we need integration by parts to evaluate the remaining integral. Set $u' = e^{-x}$ and $v = x$. We then have

$$\begin{aligned}\int x^2 e^{-x} dx &= -x^2 e^{-x} + 2\left(-xe^{-x} - \int (1)(-e^{-x}) dx\right) \\ &= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x} dx \\ &= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c\end{aligned}$$

(10) We make the substitution $u = \cos x$, then $du = -\sin x dx$ inside the integral. We have

$$\begin{aligned}\int \cos^2 x \sin x dx &= -\int u^2 du \\ &= -\frac{1}{3}u^3 + c \\ &= -\frac{1}{3}\cos^3 x + c.\end{aligned}$$

(11) When dealing with integrals in higher powers of trigonometric functions, we can use identities to simplify the expressions. In this case, we can write

$$\int \sin^3 x dx = \int \sin x \cdot \sin^2 x dx = \int \sin x (1 - \cos^2 x) dx.$$

We then make the substitution $u = \cos x$ (as in the previous example), so we also have $du = -\sin x dx$. In this case,

$$\begin{aligned}\sin^3 x dx &= -\int (1 - u^2) du \\ &= -(u - \frac{1}{3}u^3) + c \\ &= -\cos x + \frac{1}{3}\cos^3 x + c.\end{aligned}$$

(12) This is another integration by parts (see (7) above). Setting $u' = e^x$ and $v = x$ we have

$$\int xe^x dx = xe^x - \int e^x dx = (x-1)e^x + c.$$

(13) Notice that we can solve this integral easily after manipulating the integrand into a workable form:

$$\frac{2+x}{1+x} = \frac{1+1+x}{1+x} = \frac{1}{1+x} + \frac{1+x}{1+x} = \frac{1}{1+x} + 1.$$

We then have

$$\begin{aligned}\int \frac{2+x}{1+x} dx &= \int \left(\frac{1}{1+x} + 1\right) dx \\ &= \ln|1+x| + x + c.\end{aligned}$$

(14) We use the trig. substitution $x = 4\sin t$, so that $dx = 4\cos t dt$ and

$$16 - x^2 = 16 - 16\sin^2 t = 16\cos^2 t.$$

The integral becomes

$$\int \frac{dx}{\sqrt{16 - x^2}} = \int \frac{4\cos t dt}{4\cos t} = \int dt = t + c = \arcsin\left(\frac{x}{4}\right) + c.$$