DEPARTMENT OF MATHEMATICS

MATH2000
Double Integrals in Polar Coordinates and Center of Mass

---

(1) The average value of a function $f(x,y)$ over a region D in the x-y plane is

$$f_{\text{av}} = \frac{1}{A(D)} \iint_D f(x,y)dA,$$

where $A(D)$ is the area of D.

Use the above information to answer the following question.

The temperature at the point (x,y) due to a heat source at the origin is given by $T(x,y) = \dfrac{T_0}{x^2+y^2+1}$. Find the average temperature over the disc of radius a centred at the origin.

(2) Evaluate the double integral $\iint_D (x^2+y^2)dA$ where D is the region in the x-y plane bounded by a circle of radius 1 centred at the point $(1,2)$.

(3) The page and question numbers given in parentheses refer to the text (Stewart, 6 ed.).

Sketch the region whose area is given by the following integrals and evaluate the integrals.

(i) (p1014, Q5) $\displaystyle\int_{\pi}^{2\pi}\int_4^7 r\ dr\ d\theta$

(ii) (p1014, Q6) $\displaystyle\int_0^{\pi/2}\int_0^{4\cos\theta} r\ dr\ d\theta$

(4) (Stewart 6 ed., p1015, Q35) Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\ dy\ dx + \int_1^{\sqrt{2}}\int_0^x xy\ dy\ dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\ dy\ dx$$

into one double integral. Show that the integral evaluates to $\dfrac{15}{16}$.

(5) Find the coordinates of the centre of mass of the triangular lamina in the x-y plane whose vertices are $(0,0)$, $(1,0)$ and $(1,2)$ and whose density (mass per unit area) is given by the function $\rho(x,y) = k(2x - y + 1)$ (constant k).