DEPARTMENT OF MATHEMATICS

MATH2000
Double Integrals in Polar Coordinates and Centre of Mass solutions.

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(1)

$$f_{av} = \frac{1}{A(D)} \int\int_D f(x,y)dA$$
$$A(D) = \pi a^2$$

Since the region is a disc, choose polar coordinates.

$$D = \{(r,\theta) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$$

and

$$x^2 + y^2 = r^2,\ dA = r\,dr\,d\theta$$
$$\Rightarrow \quad f_{av} = \frac{1}{\pi a^2}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^a \frac{T_0 r}{r^2+1} dr\right).$$

Set $u = r^2 + 1 \Rightarrow du = 2r\,dr$

$$\begin{aligned} f_{av} &= \frac{T_0}{\pi a^2} \times 2\pi \times \frac{1}{2}\int_1^{a^2+1} \frac{du}{u} \\ &= \frac{T_0}{a^2}\left[\ \ln|u|\ \right]_1^{a^2+1} \\ &= \frac{T_0 \ln(a^2+1)}{a^2} \end{aligned}$$

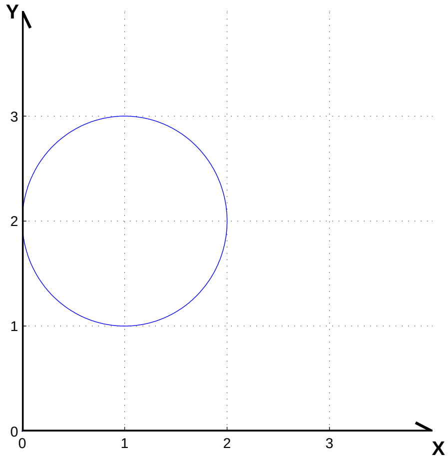
(2) Use shifted polar co-ordinates:

$$x - 1 = r\cos\theta$$
$$y - 2 = r\sin\theta$$

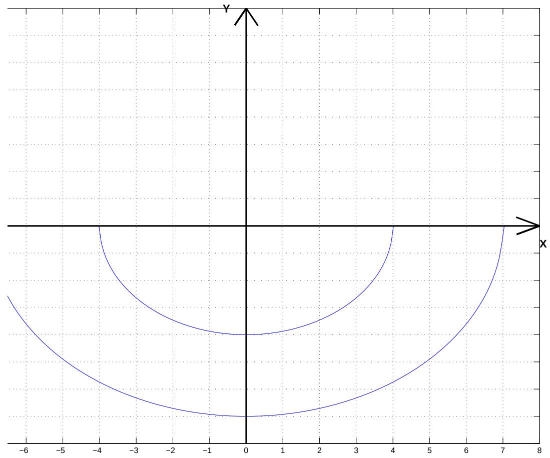
Then:

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta + 1)^2 + (r\sin\theta + 2)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta + 2r\cos\theta + 4r\sin\theta + 1 + 4 \\ &= r^2 + 2r\cos\theta + 4r\sin\theta + 5 \end{aligned}$$

$$D = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le 2\pi\}$$



$$
\begin{aligned}
\Rightarrow \iint_D (x^2+y^2)\ dx\ dy &= \int_0^1 \int_0^{2\pi} (r^2 + 2r\cos\theta + 4r\sin\theta + 5)r\ d\theta\ dr \\
&= \int_0^1 r^3 dr \int_0^{2\pi} d\theta + 2\int_0^1 r^2 dr \int_0^{2\pi} \cos\theta\ d\theta \\
&\quad +4\int_0^1 r^2 dr \int_0^{2\pi} \sin\theta\ d\theta + \int_0^1 r dr \int_0^{2\pi} d\theta \\
&= \frac{2\pi}{4} + \frac{2\times 0}{3} + \frac{4\times 0}{3} + \frac{10\pi}{2} \\
&= \frac{11\pi}{2}
\end{aligned}
$$



(3) (i)

$$\begin{aligned}
& \int_{\pi}^{2\pi}\int_{4}^{7} r dr d\theta \\
= & \int_{\pi}^{2\pi} d\theta \int_{4}^{7} r dr \\
= & \pi \times \frac{1}{2}(49-16) \\
= & \frac{33\pi}{2}
\end{aligned}$$

(ii) To draw the region, it is useful to first consider a circle of radius R, with centre at (a, b).

$$(x-a)^2 + (y-b)^2 = R^2$$

$$x = r\cos\theta, y = r\sin\theta$$

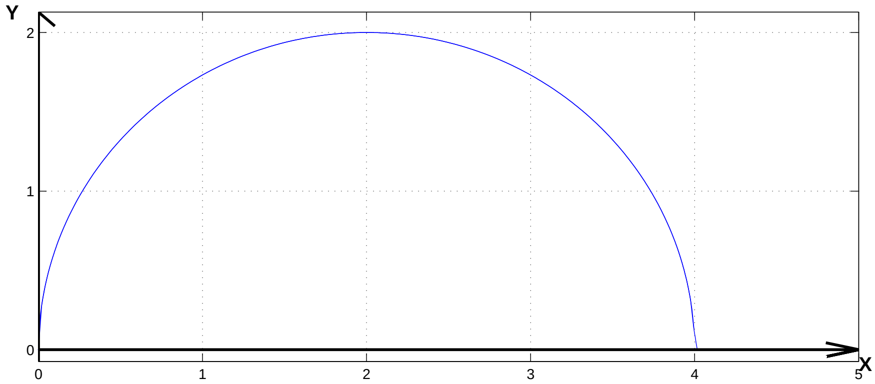
$$\Rightarrow r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta - 2br\sin\theta + b^2 = R^2$$

$$\Rightarrow r^2 = R^2 - a^2 - b^2 + 2ar\cos\theta + 2br\sin\theta$$

Now consider $r = 4\cos\theta \Rightarrow r^2 = 4r\cos\theta \rightarrow$ compare with above.

$$b = 0, R = a = 2$$

$\Rightarrow r = 4\cos\theta$ is a circle centred at $(2, 0)$ with radius 2.



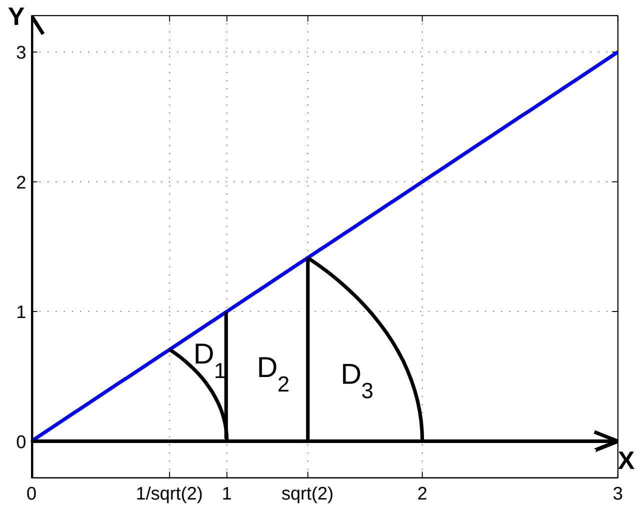
$$
\begin{aligned}
& \int_0^{\frac{\pi}{2}} \int_0^{4\cos\theta} r dr d\theta \\
= & \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_0^{4\cos\theta} d\theta \\
= & 4\int_0^{\frac{\pi}{2}} 2\cos^2\theta d\theta \\
= & 4\int_0^{\frac{\pi}{2}} (1+\cos 2\theta) d\theta \\
= & 4\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} \\
= & 2\pi
\end{aligned}
$$

(4)

$$
\int_{\frac{1}{\sqrt{2}}}^{1} \int_{\sqrt{1-x^2}}^{x} xy \ \ dy \ dx + \int_1^{\sqrt{2}} \int_0^{x} xy \ \ dy \ dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xy \ \ dy \ dx
$$

We have $D = D_1 \cup D_2 \cup D_3$, where

$$
\begin{aligned}
D_1 &= \left\{(x,y) \mid \frac{1}{\sqrt{2}} \le x \le 1, \ \sqrt{1-x^2} \le y \le x\right\} \\
D_2 &= \left\{(x,y) \mid 1 \le x \le \sqrt{2}, \ 0 \le y \le x\right\} \\
D_3 &= \left\{(x,y) \mid \sqrt{2} \le x \le 2, \ 0 \le y \le \sqrt{4-x^2}\right\}
\end{aligned}
$$



Clearly, $D = \{(r,\theta)|1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{4}\}$

$$x = r\cos\theta, y = r\sin\theta$$

$$xy = r^2 \sin\theta\cos\theta$$

$$dydx = rdrd\theta$$

So we get: $\int_1^2 \int_0^{\frac{\pi}{4}} r^3 \cos\theta \sin\theta \, d\theta dr$

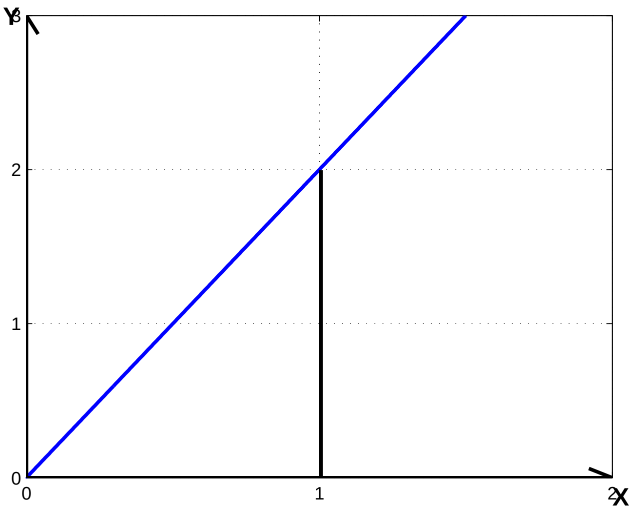
$$= \int_1^2 r^3 dr \int_0^{\frac{\pi}{4}} \cos\theta \sin\theta d\theta$$

Let $u = \sin\theta \Rightarrow du = \cos\theta$

$$= \left[\frac{r^4}{4}\right]_1^2 \int_0^{\frac{1}{\sqrt{2}}} udu$$

$$= (4 - \frac{1}{2}) \times \left[\frac{u^2}{2}\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{15}{16}$$



(5)

Centre of mass at $(\bar{x}, \bar{y})$ where:

$$\bar{x} = \frac{\iint_D x\rho(x,y)dA}{\iint_D \rho(x,y)dA}$$

$$\bar{y} = \frac{\iint_D y\rho(x,y)dA}{\iint_D \rho(x,y)dA}$$

Region is type I or II. As type I:

$$D = \{(x,y)|0 \leq x \leq 1, 0 \leq y \leq 2x\}$$

$$
\begin{aligned}
\iint_D \rho(x,y)dA &= \int_0^1 \int_0^{2x} k(2x-y+1)dydx \\
&= k\int_0^1 \left[2xy - \frac{y^2}{2} + y\right]_0^{2x} dx \\
&= k\int_0^1 2x^2 + 2xdx \\
&= 2k\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 \\
&= \frac{5k}{3} \\
\iint_D x\rho(x,y)dA &= k\int_0^1 \int_0^{2x} (2x^2 - xy + x)dydx \\
&= k\int_0^1 \left[2x^2y - \frac{xy^2}{2} + xy\right]_0^{2x} dx \\
&= k\int_0^1 2x^3 + 2x^2dx \\
&= 2k\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 \\
&= \frac{7k}{6}
\end{aligned}
$$

$$
\begin{aligned}
\iint_D y\rho(x,y)dA &= k\int_0^1 \int_0^{2x} (2xy - y^2 + y)dydx \\
&= k\int_0^1 (xy^2 - \frac{y^3}{3} + \frac{y^2}{2})dx \\
&= k\int_0^1 (\frac{4x^3}{3} + 2x^2)dx \\
&= k\left[\frac{x^4}{3} + \frac{2x^2}{3}\right]_0^1 \\
&= k \\
\Rightarrow \bar{x} &= \frac{\frac{7k}{6}}{\frac{5k}{3}} = \frac{7}{10} \\
\bar{y} &= \frac{k}{\frac{5k}{3}} = \frac{3}{5}
\end{aligned}
$$

$$\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{7}{10}, \frac{3}{5}\right)$$