DEPARTMENT OF MATHEMATICS

MATH2000
Quadratic forms (solutions)

---

(1) (a) $2x^2 - 3xy + 4y^2 - 7x + 2y + 7 = 0$

$$\Leftrightarrow \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -3/2 \\ -3/2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -7 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 7 = 0.$$

(b) $x^2 - xy + 5x + 8y - 3 = 0$

$$\Leftrightarrow \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -1/2 \\ -1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 3 = 0.$$

(c) $5xy = 8$

$$\Leftrightarrow \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 5/2 \\ 5/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 8 = 0.$$

(d) $4x^2 - 2y^2 = 7$

$$\Leftrightarrow \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 7 = 0.$$

(e) $y^2 + 7x - 8y - 5 = 0$

$$\Leftrightarrow \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 5 = 0.$$

(2) We can express the quadratic form in matrix notation by

$$\begin{aligned} Q(x,y) &= 5x^2 + 2y^2 + 4xy \\ &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \underset{\sim}{x}^T A \underset{\sim}{x}. \end{aligned}$$

We then look at orthogonally diagonalising the matrix A.

The characteristic equation of A is

$$\begin{aligned} \det(A - \lambda I) &= (5-\lambda)(2-\lambda) - 4 \\ &= 10 - 7\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda - 6)(\lambda - 1) = 0. \end{aligned}$$

For $\lambda = 6$:

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ a = 2b, \ \Rightarrow \ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ is an eigenvector,}$$

which we normalise to $\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

For $\lambda = 1$:

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ b = -2a, \ \Rightarrow \ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ is an eigenvector,}$$

which we normalise to $\begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$.

We then form the orthogonal matrix

$$P = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}, P^T = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix},$$

so that $P^{-1} = P^T$, where we have chosen the order of the columns so that $\det(P) = 1$, that is, P corresponds to a rotation (since it is orthogonal with $\det = 1$).

We then have that $A = PDP^T$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$.

The quadratic form can then be expressed as

$$Q(x, y) = \underset{\sim}{x}^T PDP^T \underset{\sim}{x}.$$

We introduce the new variables u and v such that

$$\underset{\sim}{u} = \begin{pmatrix} u \\ v \end{pmatrix} = P^T \underset{\sim}{x} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x - 2y)/\sqrt{5} \\ (2x + y)/\sqrt{5} \end{pmatrix}.$$

Hence the quadratic form can be expressed in terms of u and v:

$$Q = \underset{\sim}{u}^T D \underset{\sim}{u} = u^2 + 6v^2,$$

or rewritten in terms of x and y, we have

$$Q(x, y) = \left(\frac{x - 2y}{\sqrt{5}} \right)^2 + 6 \left(\frac{2x + y}{\sqrt{5}} \right)^2.$$

(3) The quadratic form can be expressed as

$$f(x, y) = 5x^2 + 5y^2 - 2xy = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix $\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$ has characteristic equation

$$\begin{aligned} \det \begin{pmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{pmatrix} &= 0 \\ \Rightarrow\ (5-\lambda)(5-\lambda)-1 &= 0 \\ \Rightarrow\ 25-10\lambda+\lambda^2-1 &= 0 \\ \Rightarrow\ 24-10\lambda+\lambda^2 &= 0 \\ \Rightarrow\ (6-\lambda)(4-\lambda) &= 0 \end{aligned}$$

$$\Rightarrow\ \lambda = 6, 4.$$

For $\lambda = 6$:

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow\ a = -b \ \Rightarrow\ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is an eigenvector,}$$

which we normalise to $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

For $\lambda = 4$:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow\ a = b \ \Rightarrow\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector,}$$

which we normalise to $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

We then form the orthogonal matrix

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, P^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

so that $P^{-1} = P^T$, where we have chosen the order of the columns so that $\det(P) = 1$, that is, P corresponds to a rotation (since it is orthogonal with $\det = 1$).

We then have that $A = PDP^T$, where $D = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}$.

We introduce the new variables u and v such that

$$\underset{\sim}{u} = \begin{pmatrix} u \\ v \end{pmatrix} = P^T\underset{\sim}{x} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x-y)/\sqrt{2} \\ (x+y)/\sqrt{2} \end{pmatrix}.$$

$$\Rightarrow\ 5x^2 - 2xy + 5y^2 = 6u^2 + 4v^2 = 6\left(\frac{x-y}{\sqrt{2}}\right)^2 + 4\left(\frac{x+y}{\sqrt{2}}\right)^2,$$

which is clearly > 0 when $\underset{\sim}{x} \neq \underset{\sim}{0}$.

(4) We can rewrite the equation as

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -14.$$

We seek to orthogonally diagonalise $A = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}$, in order to describe the rotation of the coordinate axes. To this end,

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-1-\lambda) - 4 \\ &= -2 - \lambda + \lambda^2 - 4 \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda - 3)(\lambda + 2) = 0. \end{aligned}$$

For $\lambda = -2$:

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow b = 2a \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is an eigenvector,}$$

which we normalise to $\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

For $\lambda = 3$:

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -2b \Rightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ is an eigenvector,}$$

which we normalise to $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

We then form the orthogonal matrix

$$P = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}, P^T = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix},$$

so that $P^{-1} = P^T$, where we have chosen the order of the columns so that $\det(P) = 1$, that is, P corresponds to a rotation (since it is orthogonal with det $= 1$).

We then have that $A = PDP^T$, where $D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$.

We introduce the new variables u and v such that

$$\underset{\sim}{u} = \begin{pmatrix} u \\ v \end{pmatrix} = P^T\underset{\sim}{x} = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x+2y)/\sqrt{5} \\ (-2x+y)/\sqrt{5} \end{pmatrix}.$$

It is also useful to note the reverse rotation, so that

$$\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix} = P\underset{\sim}{u} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (u-2v)/\sqrt{5} \\ (2u+v)/\sqrt{5} \end{pmatrix}.$$

The equation can then be written in terms of the new variables as

$$\begin{pmatrix} u & v \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -4 & -8 \end{pmatrix}\begin{pmatrix} (u-2v)/\sqrt{5} \\ (2u+v)/\sqrt{5} \end{pmatrix} = -14$$

$$\Rightarrow\ -2u^2 + 3v^2 - \frac{20}{\sqrt{5}}u = -14.$$

To finish the process of writing the equation in standard form, we complete the square in the variable u:

$$\Rightarrow\ -2\left(u^2 + \frac{10}{\sqrt{5}}u + 5\right) + 10 + 3v^2 = -14$$

$$\Rightarrow\ -2\left(u + \frac{5}{\sqrt{5}}\right)^2 + 3v^2 = -24.$$

Hence we arrive at the final set of variables s and t, given by

$$\begin{aligned}\begin{pmatrix} s \\ t \end{pmatrix} &= \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 5/\sqrt{5} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5/\sqrt{5} \\ 0 \end{pmatrix},\end{aligned}$$

or equivalently

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} - \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 5/\sqrt{5} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}.\end{aligned}$$

The equation in terms of s and t can be expressed as

$$\frac{s^2}{12} - \frac{t^2}{8} = 1.$$

This is the equation of a hyperbola.

Question: are you able to plot the graph?

(5) We can rewrite the equation as

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -10 & -20 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 5 = 0.$$

We seek to orthogonally diagonalise $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$, in order to describe the rotation of the coordinate axes. To this end,

$$\begin{aligned}\det(A - \lambda I) &= (9-\lambda)(6-\lambda) - 4 \\ &= 54 - 15\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 15\lambda + 50 \\ &= (\lambda - 10)(\lambda - 5) = 0.\end{aligned}$$

For $\lambda = 5$:

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow b = 2a \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is an eigenvector,}$$

which we normalise to $\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

For $\lambda = 10$:

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -2b \Rightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ is an eigenvector,}$$

which we normalise to $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

We then form the orthogonal matrix

$$P = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}, P^T = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix},$$

so that $P^{-1} = P^T$, where we have chosen the order of the columns so that $\det(P) = 1$, that is, P corresponds to a rotation (since it is orthogonal with $\det = 1$).

We then have that $A = PDP^T$, where $D = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}$.

We introduce the new variables u and v such that

$$\underset{\sim}{u} = \begin{pmatrix} u \\ v \end{pmatrix} = P^T \underset{\sim}{x} = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x+2y)/\sqrt{5} \\ (-2x+y)/\sqrt{5} \end{pmatrix}.$$

It is also useful to note the reverse rotation, so that

$$\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix} = P\underset{\sim}{u} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (u-2v)/\sqrt{5} \\ (2u+v)/\sqrt{5} \end{pmatrix}.$$

The equation can then be written in terms of the new variables as

$$\begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -10 & -20 \end{pmatrix} \begin{pmatrix} (u-2v)/\sqrt{5} \\ (2u+v)/\sqrt{5} \end{pmatrix} - 5 = 0$$

$$\Rightarrow 5u^2 + 10v^2 - 10\sqrt{5}u - 5 = 0.$$

To finish the process of writing the equation in standard form, we complete the square in the variable u:

$$\Rightarrow 5\left(u^2 - 2\sqrt{5}u + 5\right) - 25 + 10v^2 - 5 = 0$$

$$\Rightarrow 5\left(u - \sqrt{5}\right)^2 + 10v^2 = 30.$$

Hence we arrive at the final set of variables s and t, given by

$$\begin{aligned}
\begin{pmatrix} s \\ t \end{pmatrix} &= \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -\sqrt{5} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\sqrt{5} \\ 0 \end{pmatrix},
\end{aligned}$$

or equivalently

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} - \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} -\sqrt{5} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\end{aligned}$$

The equation in terms of s and t can be expressed as

$$\frac{s^2}{6} + \frac{t^2}{3} = 1.$$

This is the equation of an ellipse.