DEPARTMENT OF MATHEMATICS

MATH2000
Nonhomogeneous Linear Second Order ODEs solutions.

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(1) To find y_H use $y_H = e^{mx}$ so $m^2 + m - 2 = (m+2)(m-1) = 0$ so $y_H = c_1 e^{-2x} + c_2 e^x$. To find y_p use table to select $y = A\sin x + B\cos x$, so $y' = A\cos x - B\sin x$ and $y'' = -A\sin x - B\cos x$. Sub into equation

$$(-3B + A)\cos x + (-3A - B)\sin x = 10\cos x + 0 \times \sin x.$$

Which leads to (i) -3B+A=10 and (ii) -3A-B=0. Now $3\times$ (i) + (ii) gives $-10B = 30$ so $B = -3$ and (i) gives $3A = -B$ so $A = 1$. Thus $y_p = \sin x - 3\cos x$ and the general solution is

$$y = y_h + y_p = c_1 e^{-2x} + c_2 e^x + \sin x - 3\cos x.$$

(2) $y_H = e^{mx}$ gives $m^2 - 3m + 2 = (m-1)(m-2) = 0$ so $y_H = c_1 e^x + c_2 e^{2x}$. Now $r(x)$ is part of y_H so try $y_p = Axe^{2x}$. Now $y_p' = Ae^{2x} + Axe^{2x}$ and $y_p'' = 2Ae^{2x} + 2Ae^{2x} + 2Axe^{2x}$ Sub into equation gives

$$A4xe^{2x} + 4e^{2x} - 3(Ae^{2x} + A2xe^{2x}) + 2Axe^{2x} = Ae^{2x} = e^{2x}$$

so $A = 1$ and general solution is $y = y_H + y_p = c_1 e^x + c_2 e^{2x} + xe^{2x}$. Now $y(0) = c_1 + c_2 = 2$, and $y'(x) = c_1 e^x + 2c_2 e^{2x} + e^{2x} + xe^{2x}$ so $y'(0) = 0 = c_1 + 2c_2 + 1 = 0$. Solving gives $c_2 = -3$ and $c_1 = 5$ so

$$y = 5e^x - 3e^{2x} + xe^{2x}.$$

(3) Find $y_h = e^{mx}$. Find $m^2 + 2m = m(m+2) = 0$ so $y_h = c_1 e^{0x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$. Note the RHS=1 is part of y_h so try $y_p = Ax \times 1 + B$ (the B is not needed). $y_p' = A$ and $y_p'' = 0$ which means y_p is a solution if $2A = 1$ or $A = \frac{1}{2}$. Thus the general solutions is

$$y = c_1 + c_2 e^{-2x} + \frac{1}{2}x.$$

Because y does not appear in the original equation you let $w = y'$ and the equation becomes first-order linear in w.

(4) To solve $y'' - y' - 6y = 1 + 6x^2$, first solve the homogeneous equation $y'' - y' - 6y = 0$ for y_H, then use the method of undetermined coefficients to find y_P. The general solution will be $y = y_H + y_P$.

The equation $y'' - y' - 6y = 0$ has characteristic equation

$$\begin{aligned} \lambda^2 - \lambda - 6 &= 0 \\ \Rightarrow\ (\lambda - 3)(\lambda + 2) &= 0 \\ \Rightarrow\ \lambda &= 3, -2 \\ \Rightarrow\ y_H &= Ae^{3x} + Be^{-2x}. \end{aligned}$$

For y_P, we make the guess

$$\begin{aligned} y_P &= a + bx + cx^2 \\ \Rightarrow\ y_P' &= b + 2cx \\ \Rightarrow\ y_P'' &= 2c. \end{aligned}$$

This guess is based on the form of the right hand side of the ODE. Substituting into the nonhomogeneous ODE gives

$$\begin{aligned} y_P'' - y_P' - 6y_P &= 2c - b - 2cx - 6a - 6bx - 6cx^2 \\ &= 1 + 6x^2. \end{aligned}$$

Equating the coefficients of 1, x, x^2 gives

$$\begin{aligned} -6c &= 6, \\ -2c - 6b &= 0, \\ 2c - b - 6a &= 1. \end{aligned}$$

Back substitution then gives $c = -1 \Rightarrow b = 1/3 \Rightarrow a = -5/9$. Therefore the general solution is

$$y(x) = y_H + y_P = Ae^{3x} + Be^{-2x} - \frac{5}{9} + \frac{1}{3}x - x^2.$$

The first derivative is

$$y' = 3Ae^{3x} - 2Be^{-2x} + \frac{1}{3} - 2x.$$

We now impose the inital conditions

$$\begin{aligned} y(0) &= \frac{1}{3} = A + B - \frac{5}{9} \\ y'(0) &= \frac{4}{3} = 3A - 2B + \frac{1}{3} \end{aligned}$$

which yields $A = 5/9$, $B = 1/3$. The solution to the I.V.P. is then

$$y(x) = \frac{5}{9}e^{3x} + \frac{1}{3}e^{-2x} - \frac{5}{9} + \frac{1}{3}x - x^2.$$

(5) The general solution is of the form $y = y_H + y_P$ where y_H is the general solution to $y'' + y = 0$. This has characteristic equation

$$\lambda^2 + 1 = 0 \ \Rightarrow\ \lambda = \pm i$$

$$\Rightarrow\ y_H = A\cos x + B\sin x.$$

Since $\sin x$ appears on the right hand side of the ODE *and* in y_H, we guess that y_P is of the form

$$\begin{aligned} y_P &= ax\cos x + bx\sin x \\ \Rightarrow\ y_P' &= a\cos x - ax\sin x + b\sin x + bx\cos x \\ \Rightarrow\ y_P'' &= -a\sin x - a\sin x - ax\cos x + b\cos x + b\cos x - bx\sin x. \end{aligned}$$

Now substitute this into $y'' + y = \sin x$

$$\begin{aligned} \Rightarrow \ -2a\sin x - ax\cos x + 2b\cos x - bx\sin x + ax\cos x + bx\sin x &= \sin x \\ \Rightarrow \ -2a\sin x + 2b\cos x = \sin x & \\ \Rightarrow \ a = \frac{1}{2}, \quad b = 0 & \\ \Rightarrow \ y_P = -\frac{1}{2}x\cos x & \end{aligned}$$

Therefore the general solution is

$$y(x) = y_H + y_P = A\cos x + B\sin x - \frac{1}{2}x\cos x.$$

To impose the initial conditions, we need the first derivative which is

$$y' = -A\sin x + B\cos x - \frac{1}{2}\cos x + \frac{1}{2}x\sin x$$

$$\begin{aligned} \Rightarrow \ y(0) = 1 &= A \\ \text{and } y'(0) = 0 &= B - \frac{1}{2} \ \Rightarrow \ B = \frac{1}{2}. \end{aligned}$$

Therefore the solution to the I.V.P. is

$$y(x) = \cos x + \frac{1}{2}\sin x - \frac{1}{2}x\cos x.$$

(6) The homogeneous equation is $i'' + 4i' + 8i = 0$, with $i = e^{mt}$ we have $m^2 + 4m + 8 = 0$, now $b^2 - 4ac = -16 < 0$ so $m = -2 \pm 2i$. So

$$i_h = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)$$

(a) Since RHS=0 the general solution is $i(t) = i_h \to 0$ as $t \to 0$.

(b) We try the solution $i_p = A$, which gives us $8A = 1$ or $A = 1/8$. Thus

$$i(t) = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t) + \frac{1}{8}.$$

As $t \to \infty$ $i(t) \to 1/8$.

(c) We try the solution $i_p = A\cos(2t) + B\sin(2t)$ so $i_p' = -2A\sin(2t) + 2B\cos(2t)$, $i_p'' = -4A\cos(2t) - 4B\sin(2t)$. Sub into equation

$$(-4A + 8B + 8A)\cos(2t) + (-4B - 8A + 8B)\sin(2t) = 20\cos(2t) + 0\sin(2t).$$

Solving the second equation gives $B = 2A$, which gives from the first equation $16A + 4A = 20$ or $A = 1$.

$$i(t) = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t) + \cos(2t) + 2\sin(2t)$$

As $t \to \infty$ we have

$$i(t) = \cos(2t) + 2\sin(2t)$$

which is the steady state current.

(7) Solution will be of the form $y = y_H + y_P$, where y_H solves the homogeneous part ($y'' + 4y = 0$) and y_P is the particular solution. y_H has the characteristic equation:

$$\lambda^2 + 4 = 0$$

$$\Rightarrow \lambda = 0 \pm 2i$$

$$\Rightarrow y_H = A\cos 2x + B\sin 2x$$

Since $\cos 2x$ appears in y_H and the right hand side of the ODE, we guess that the solution y_P is of the form

$$y_P = ax\cos 2x + bx\sin 2x$$

$$y_P' = a\cos 2x - 2ax\sin 2x + b\sin 2x + 2bx\cos 2x$$

$$y_P'' = -4a\sin 2x - 4ax\cos 2x + 4b\cos 2x - 4bx\sin 2x$$

Substituting this into the ODE:

$$-4a\sin 2x + 4b\cos 2x = \cos 2x$$

$$\Rightarrow a = 0, b = \frac{1}{4}$$

$$\Rightarrow y_P = \frac{1}{4}x\sin 2x$$

$$\Rightarrow y = A\cos 2x + B\sin 2x + \frac{1}{4}x\sin 2x$$

Now using the initial conditions:

$$y(0) = 2 = A$$

$$y' = -2A\sin 2x + 2B\cos 2x + \frac{1}{4}\sin 2x + \frac{1}{2}x\cos 2x$$

$$y'(0) = 2 = 2B$$

$$\Rightarrow y(x) = 2\cos 2x + \sin 2x + \frac{1}{4}x\sin 2x$$

(8) Solution will be of the form $y = y_H + y_P$, where y_H solves the homogenous part ($y'' - 6y' + 9y = 0$) and y_P is the particular solution. y_H has the characteristic equation:

$$(\lambda - 3)^2 = 0$$

$$\Rightarrow \lambda = 3$$

$$\Rightarrow y_H = Ae^{3x} + Bxe^{3x}$$

Since the double root $\lambda = 3$ appears in y_H, we guess that y_P has the form

$$y_P = ax^2e^{3x} + b\cos x + c\sin x$$

$$y_P' = 2axe^{3x} + 3ax^2e^{3x} - b\sin x + c\cos x$$

$$y_P'' = 2ae^{3x} + 12axe^{3x} + 9ax^2e^{3x} - b\cos x - c\sin x$$

Substituting this into the ODE:

$$2ae^{3x} + (8b - 6c)\cos x + (8c + 6b)\sin x = 4e^{3x} + 14\cos x$$

$$\Rightarrow \quad a = 2, b = \frac{28}{25}, c = -\frac{21}{25}$$

$$y(x) = Ae^{3x} + Bxe^{3x} + 2x^2e^{3x} + \frac{28}{25}\cos x - \frac{21}{25}\sin x$$

$$y'(x) = 3Ae^{3x} + Be^{3x} + 3Bxe^{3x} + 4xe^{3x} + 6x^2e^{3x} - \frac{28}{25}\sin x - \frac{21}{25}\cos x$$

$$y(0) = A + \frac{28}{25} = 4 \Rightarrow \quad A = \frac{72}{25}$$

$$y'(0) = 3A + B - \frac{21}{25} = 5 \Rightarrow B = -\frac{14}{5}$$

$$y(x) = \frac{72}{25}e^{3x} - \frac{14}{5}xe^{3x} + 2x^2e^{3x} + \frac{28}{25}\cos x - \frac{21}{25}\sin x$$

(9) Assuming $y_H = x^\lambda$, we get:

$$\lambda(\lambda - 1)x^\lambda + \lambda x^\lambda - n^2x^\lambda = 0$$

$$\lambda^2 - n^2 = 0$$

$$\Rightarrow \lambda = \pm n$$

$$\Rightarrow \quad y_1 = x^n, \quad y_2 = x^{-n}$$

$$\Rightarrow \quad y_H = Ax^n + \frac{B}{x^n}$$

Using variation of parameters, $r = x^{m-2}$ and $W = y_1y_2' - y_2y_1' = -\frac{2n}{x}$, so:

$$\begin{aligned} u &= -\int \frac{y_2 r}{W}dx \quad \text{(lectures)} \\ &= \int \frac{x^{-n}x^{m-2}x}{2n}dx \\ &= \frac{1}{2n}\int x^{m-n-1}dx \\ &= \frac{x^{m-n}}{2n(m-n)} \end{aligned}$$

$$
\begin{aligned}
v &= -\int \frac{y_1 r}{W} dx \quad \text{(lectures)} \\
&= \int \frac{x^n x^{m-2} x}{2n} dx \\
&= \frac{1}{2n} \int x^{m+n-1} dx \\
&= \frac{x^{m+n}}{2n(m+n)} \\
\Rightarrow y_p &= u y_1 + v y_2 \\
&= \frac{x^{m-n} x^n}{2n(m-n)} - \frac{x^{m+n} x^{-n}}{2n(m+n)} \\
&= \frac{x^m}{m^2 - n^2}
\end{aligned}
$$

So now we have $y = y_H + y_P$

$$y = Ax^n + \frac{B}{x^n} + \frac{x^m}{m^2 - n^2}$$

(10) Solution will be of the form $y = y_H + y_P$, where y_H solves the homogenous part ($y'' - 2y' + y = 0$) and y_P is the particular solution. y_H has the characteristic equation:

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = 1 \ \text{(double root)}$$

$$\Rightarrow y_H = Ae^x + Bxe^x$$

To find y_P, we use variation of parameters, where we set $y_P = u(x)y_1 + v(x)y_2$. From lectures, Wronksian, W, is:

$$y_1 y_2' - y_2 y_1' = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

And using the formulae for u and v:

$$
\begin{aligned}
u &= -\int \frac{y_2 r}{W} dx \\
&= -\int \frac{xe^x e^x}{e^{2x} x^3} dx \\
&= -\int \frac{1}{x^2} dx \\
&= \frac{1}{x} \\
v &= \int \frac{e^x e^x}{e^{2x} x^3} dx \\
&= -\int \frac{1}{x^3} dx \\
&= -\frac{1}{2x^2}
\end{aligned}
$$

$$\Rightarrow y_P = (\frac{1}{x})e^x + (-\frac{1}{2x^2})xe^x$$
$$= \frac{e^x}{2x}$$

So the general solution is

$$y = Ae^x + Bxe^x + \frac{e^x}{2x}.$$

(11) Since RHS is not of a simple form use method of V of P's. Try $y_h = e^{mx}$ which gives $m^2 - 2m + 1 = (m-1)^2 = 0$ so $y_1 = e^x$ and $y_2 = xe^x$. Find u_1 and u_2 such that $y_p = u_1 y_1 + u_2 y_2$ satisfies the equation. Note $W(x) = y_1 y_2' - y_1' y_2 = e^x(e^x + xe^x) - xe^x e^x = e^{2x}$ so

$$u_1 = -\int \frac{y_2 r(x)}{W(x)} dx = -\int \frac{xe^x x^{\frac{3}{2}} e^x}{e^{2x}} dx = -\int x^{\frac{5}{2}} dx = -\frac{2}{7} x^{\frac{7}{2}} + c_1$$
$$u_2 = \int \frac{y_1 r(x)}{W(x)} dx = \int \frac{e^x x^{\frac{3}{2}} e^x}{e^{2x}} dx = \int x^{\frac{3}{2}} dx = \frac{2}{5} x^{\frac{5}{2}} + c_2$$

So

$$y_p = u_1 y_1 + u_2 y_2 = e^x(-\frac{2}{7} x^{\frac{7}{2}} + c_1) + xe^x(\frac{2}{5} x^{\frac{5}{2}} + c_2) = \frac{4}{35} x^{\frac{7}{2}} e^x + c_1 y_1 + c_2 y_2,$$

which is the general solution since we incorporated the constants c_i.

(12) For y_H, solve $y'' + y = 0$. This has characteristic equation $\lambda^2 + 1 = 0$, so that $\lambda = \pm i$. Therefore,

$$y_H = A\cos x + B\sin x.$$

We use variation of parameters, with

$$\begin{aligned}
y_1 &= \cos x, \quad y_2 = \sin x \\
r &= \cot x = \frac{\cos x}{\sin x} \\
W = y_1 y_2' - y_1' y_2 &= \cos x \cos x - (-\sin x)\sin x = 1. \\
\Rightarrow \quad u &= -\int \frac{y_2 r}{W} dx = -\int \sin x \cdot \frac{\cos x}{\sin x} dx \\
&= -\cos x dx = -\sin x, \\
\text{and } \ v &= \int \frac{y_2 r}{W} dx = \int \cos x \cdot \frac{\cos x}{\sin x} dx \\
&= \int \frac{\cos^2 x}{\sin x} dx \\
&= \int \frac{1 - \sin^2 x}{\sin x} dx \\
&= \int \frac{dx}{\sin x} - \int \sin x dx \\
&= \int \frac{\sin x}{\sin^2 x} dx + \cos x \\
&= \int \frac{\sin x}{1 - \cos^2 x} dx + \cos x.
\end{aligned}$$

We make the substitution $t = \cos x \Rightarrow$ in the integral $dt = -\sin x dx$

$$\begin{aligned}
\Rightarrow \quad v &= -\int \frac{dt}{1-t^2} + \cos x \\
&= -\frac{1}{2}\int \frac{dt}{1-t} - \frac{1}{2}\int \frac{dt}{1+t} + \cos x \text{ (partial fractions)} \\
&= \frac{1}{2}\ln|1-t| - \frac{1}{2}\ln|1+t| + \cos x \\
&= \frac{1}{2}\ln\left|\frac{1-t}{1+t}\right| + \cos x \\
&= \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + \cos x \\
\Rightarrow \quad y_P &= -\cos x \sin x + \frac{1}{2}\sin x \ln\left|\frac{1-\cos x}{1+\cos x}\right| + \sin x \cos x \\
&= \frac{1}{2}\sin x \ln\left|\frac{1-\cos x}{1+\cos x}\right|.
\end{aligned}$$

The general solution is then

$$y = A\cos x + B\sin x + \frac{1}{2}\sin x \ln\left|\frac{1-\cos x}{1+\cos x}\right|.$$