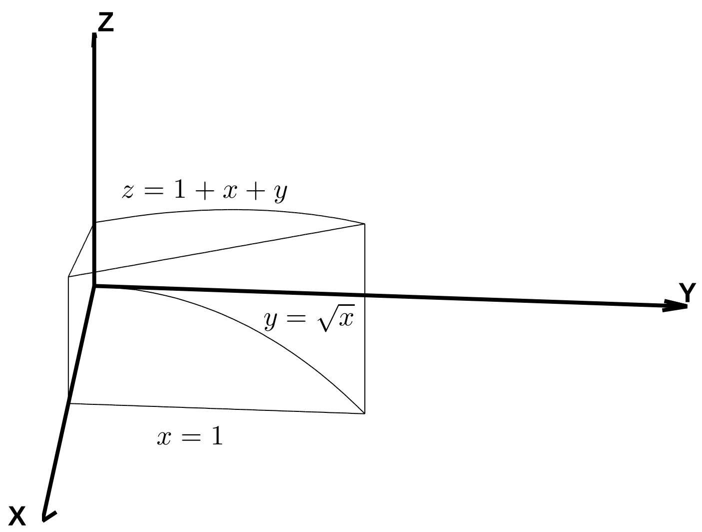
DEPARTMENT OF MATHEMATICS

MATH2000
Triple Integrals in Rectangular Coordinates solutions.

(1)

$$\begin{aligned}
\int_0^2 \int_0^y \int_0^{\sqrt{4-y^2}} 2x\ dx\ dz\ dy &= \int_0^2 \int_0^y \left[x^2\right]_0^{\sqrt{4-y^2}}\ dz\ dy \\
&= \int_0^2 \int_0^y (4-y^2)\ dz\ dy \\
&= \int_0^2 \left[4z - y^2 z\right]_{z=0}^{z=y} dy \\
&= \int_0^2 (4y - y^3) dy \\
&= \left[2y^2 - \frac{1}{4}y^4\right]_0^2 \\
&= 8 - 4 = 4.
\end{aligned}$$

(2) Region of integration is defined to lie above the region in the x-y plane bounded by $y = \sqrt{x}$, $y = 0$ and $x = 1$:

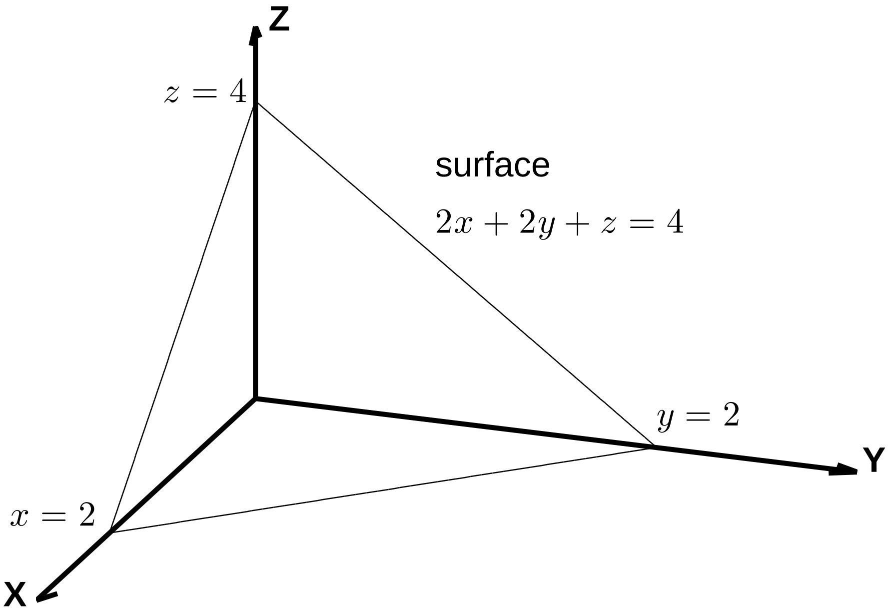


Region of integration D in $3D$ is above this region in x-y plane and below surface $z = 1 + x + y$:

$$D = \{(x, y, z) | 0 \le x \le 1,\ 0 \le y \le \sqrt{x},\ 0 \le z \le 1 + x + y\}$$

$$
\begin{aligned}
\Rightarrow \quad \iiint_D 6xy\ dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\ dz\ dy\ dx \\
&= \int_0^1 \int_0^{\sqrt{x}} [6xyz]_0^{1+x+y}\ dy\ dx \\
&= 6 \int_0^1 \int_0^{\sqrt{x}} (xy + x^2y + xy^2)\ dy\ dx \\
&= 6 \int_0^1 \left[\frac{1}{2}xy^2 + \frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_0^{\sqrt{x}}\ dx \\
&= 6 \int_0^1 \left(\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^{5/2}\right)\ dx \\
&= 6 \left[\frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{2}{21}x^{7/2}\right]_0^1 \\
&= 6 \left(\frac{1}{6} + \frac{1}{8} + \frac{2}{21}\right) = \frac{65}{28}.
\end{aligned}
$$

(3)

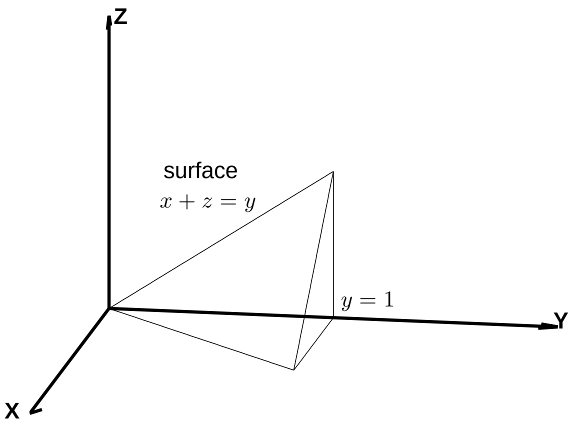


The region of integration is:

$$D = \{(x, y, z) | 0 \le x \le 2,\ 0 \le y \le 2 - x,\ 0 \le z \le 4 - 2x - 2y\}$$

$$
\begin{aligned}
\iiint_D y\, dV &= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y\, dz\, dy\, dx \\
&= \int_0^2 \int_0^{2-x} [yz]_{z=0}^{z=4-2x-2y}\, dy\, dx \\
&= \int_0^2 \int_0^{2-x} \left(4y - 2xy - 2y^2\right)\, dy\, dx \\
&= \int_0^2 \left[2y^2 - xy^2 - \frac{2}{3}y^3\right]_{y=0}^{y=2-x} dx \\
&= \int_0^2 \left(2(2-x)^2 - x(2-x)^2 - \frac{2}{3}(2-x)^3\right) dx \\
&= \int_0^2 \frac{1}{3}(2-x)^3 dx \\
&= \left[-\frac{1}{12}(2-x)^4\right]_0^2 \\
&= 0 - \left(-\frac{16}{12}\right) \\
&= \frac{4}{3}.
\end{aligned}
$$

(4) The challenge here is to first work out the equations of the intersecting planes.



Clearly 2 of the 4 intersecting planes are the coordinate planes $x = 0$ and $z = 0$. There is also the plane $y = 1$. For the 4th, we can work out from the diagram the equations for the 3 lines where the 4th plane intersects with the other 3 planes. They are:

$(x = 0,\ y = z)$
$(y = 1,\ x + z = 1)$

$(z = 0,\ y = x)$

Combining these gives the equation of the plane $x + z = y$.

The region of integration can be expressed as

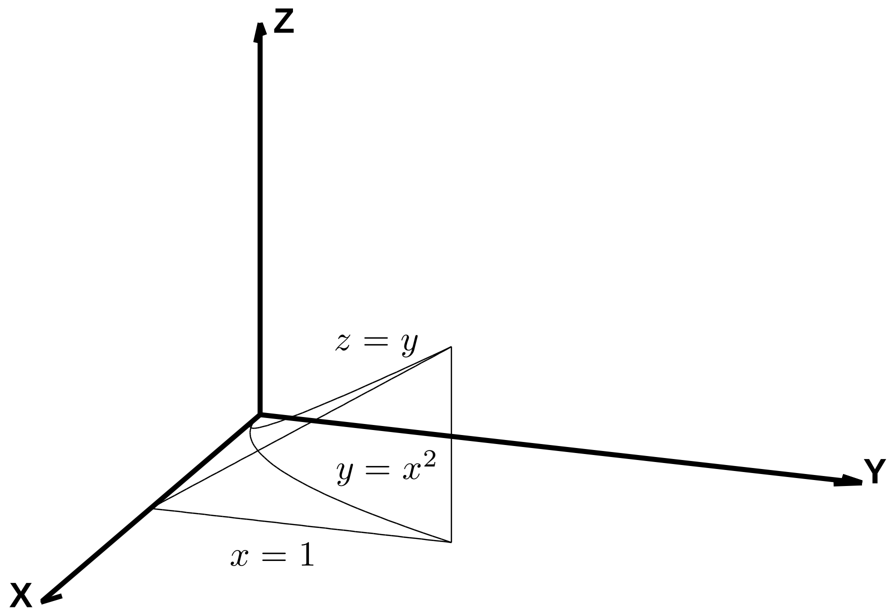
$$D = \{(x, y, z) | 0 \le x \le 1,\ x \le y \le 1,\ 0 \le z \le y - x\}.$$

$$\begin{aligned}
\iiint_D xz\ dV &= \int_0^1 \int_x^1 \int_0^{y-x} xz\ dz\ dy\ dx \\
&= \int_0^1 \int_x^1 \left[\frac{1}{2}xz^2\right]_{z=0}^{z=y-x} dy\ dx \\
&= \int_0^1 \int_x^1 \frac{1}{2}x(y^2 - 2xy + x^2)\ dy\ dx \\
&= \int_0^1 \left[\frac{1}{6}xy^3 - \frac{1}{2}x^2y^2 + \frac{1}{2}x^3y\right]_{y=x}^{y=1} dx \\
&= \int_0^1 (\frac{1}{6}x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4)\ dx \\
&= \frac{1}{12} - \frac{1}{6} + \frac{1}{8} - \frac{1}{30} = \frac{1}{120}
\end{aligned}$$

(5) The region of integration can be expressed as

$$\left\{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x^2,\ 0 \leq z \leq y\right\}.$$

From this information you might be able to draw a diagram of the region and then write down the other five integrals straight off. This is a very efficient way of doing it, but it is certainly not easy.

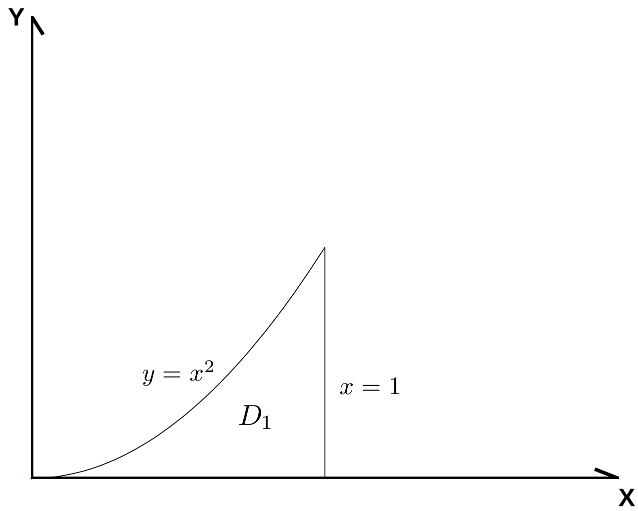


A more straightforward and systematic approach is to swap the order of integration of two variables at a time. This is equivalent to changing the order of a double integral for each pair of variables.

$$
\begin{aligned}
I &= \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\ dz\ dy\ dx \quad (1) \\
&= \iint_{D_1} \left(\int_0^y f\ dz \right)\ dA_{xy}
\end{aligned}
$$

where D_1 is the region in the x-y plane

$$D_1 = \{(x,y)|0 \le x \le 1,\ 0 \le y \le x^2\}$$



To change the order of x and y, we can also represent the region as

$$D_1 = \{(x,y)|0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$$

$$\Rightarrow \quad I = \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f\ dz\ dx\ dy. \quad (2)$$

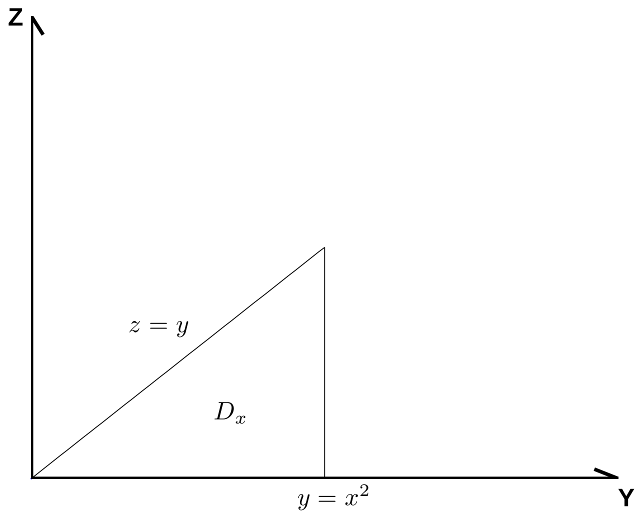
Also from (1),

$$I = \int_0^1 \left(\iint_{D_x} f \; dA_{yz} \right) \; dx$$

where the double integral is over a region in a plane parallel to the y-z plane for any $x \in [0, 1]$. Effectively we treat x as a constant. The region can be expressed as

$$D_x = \{(y, z) | 0 \le y \le x^2, \; 0 \le z \le y\}.$$

The region D_x looks like this:



We can also express the region in the other order as

$$D_x = \{(y, z) | 0 \le z \le x^2, \; z \le y \le x^2\}$$

$$\Rightarrow \quad I = \int_0^1 \int_0^{x^2} \int_z^{x^2} f \; dy \; dz \; dx \quad (3)$$

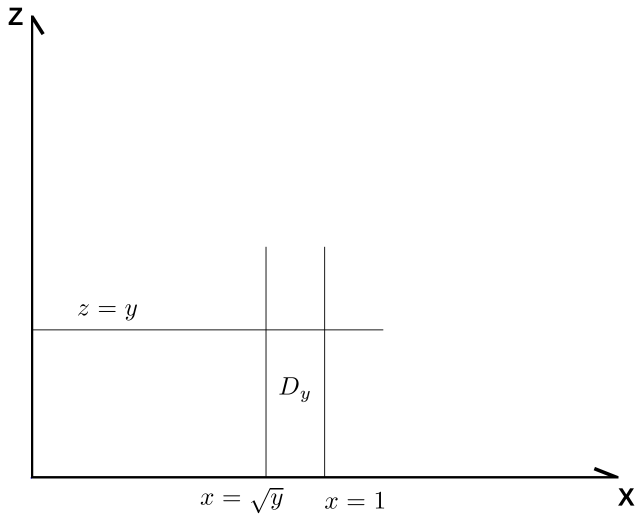
From (2),

$$I = \int_0^1 \left(\iint_{D_y} f \; dA_{xz} \right) \; dy$$

where the double integral is over a region in a plane parallel to the x-z plane for any $y \in [0,1]$. So effectively we treat y as a constant:

$$D_y = \{(x,z) | \sqrt{y} \leq x \leq 1, \; 0 \leq z \leq y\}.$$

For any y satisfying $0 \leq y \leq 1$, the region D_y looks like this:



We can change order trivially in this case, so that

$$I = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f \; dx \; dz \; dy. \quad (4)$$

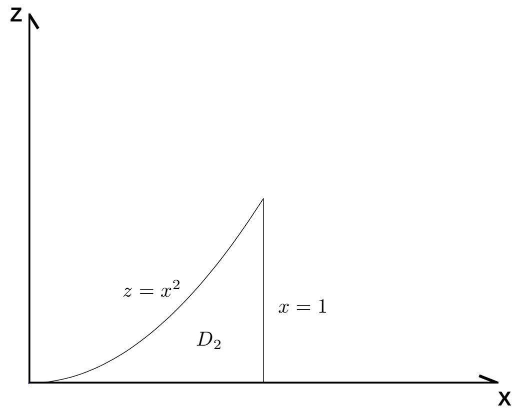
From (3),

$$I = \iint_{D_2} \left(\int_z^{x^2} f \; dy \right) \; dA_{xz}$$

where D_2 is the region in the x-z plane

$$D_2 = \{(x,z) | 0 \leq x \leq 1, \; 0 \leq z \leq x^2\}.$$

The region D_2 looks like this:



It can be expressed in the other order, from which we read the bounds of the new integral:

$$D_2 = \{(x, z) | 0 \leq z \leq 1, \ \sqrt{z} \leq x \leq 1\}$$

$$\Rightarrow \quad \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f \ dy \ dx \ dz. \quad (5)$$

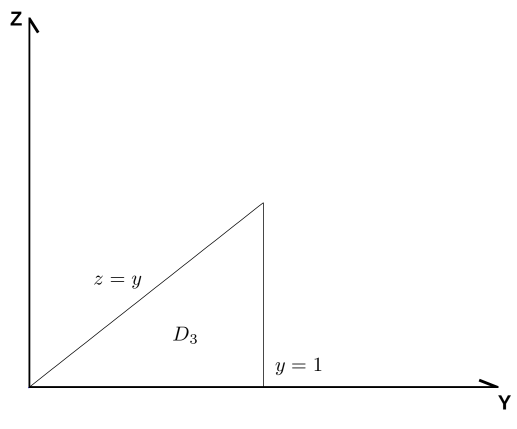
Finally, from (4),

$$I = \iint_{D_3} \left(\int_{\sqrt{y}}^1 f \ dx \right) \ dA_{yz}$$

where D_3 is the region in the y-z plane

$$D_3 = \{(y, z) | 0 \leq y \leq 1, \ 0 \leq z \leq y\}.$$

In the y-z plane, D_3 looks like this:



To swap the order of integration, we write

$$D_3 = \{(y,z) | 0 \le z \le 1,\ z \le y \le 1\}$$

$$\Rightarrow \quad I = \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f \ dx \ dy \ dz. \quad (6)$$

The six orders of integration are given by (1), (2), (3), (4), (5) and (6).