Lec. 11: MATH2100/2010

Another example:

$$\begin{cases} y_1'(t) = y_1(t) + y_2(t)^2 \\ y_2'(t) = y_2(t) + y_1(t)^2 \end{cases} \qquad (11.1)$$

$$F_1(y_1, y_2) = y_1 + y_2^2 \,, \qquad F_2(y_1, y_2) = y_2 + y_1^2$$

Critical points:

$F_1 = 0 \Rightarrow y_1 + y_2^2 = 0 \Rightarrow y_1 = -y_2^2$

$F_2 = 0 \Rightarrow y_2 + y_1^2 = 0 \rightarrow y_2 + y_2^4 = 0$

$\Rightarrow y_2(1 + y_2^3) = 0$

$\Rightarrow y_2 = 0$ or -1

when $y_2 = 0$, $\quad y_1 + y_2^2 = 0 \Rightarrow y_1 = 0$

when $y_2 = -1$, $\quad y_1 + y_2^2 = 0 \Rightarrow y_1 = -1$

So critical points are: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

[or $(0,0)$, $(-1,-1)$]

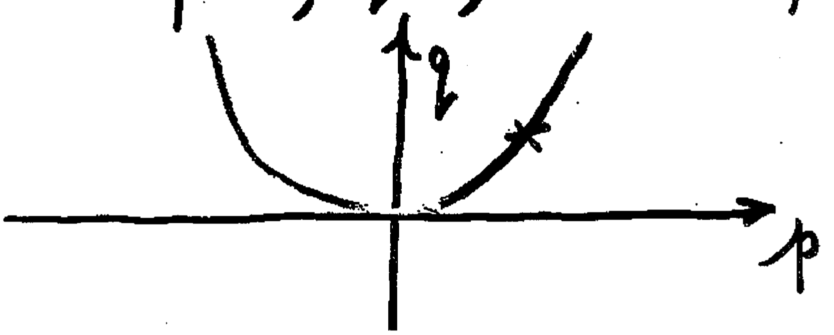
Next:
$$F_1(y_1, y_2) = y_1 + y_2^2$$
$$F_2(y_1, y_2) = y_2 + y_1^2$$

$$\Rightarrow \quad F_{11}(y_1, y_2) = \frac{\partial F_1(y_1, y_2)}{\partial y_1} = 1$$
$$F_{12}(y_1, y_2) = \frac{\partial F_1(y_1, y_2)}{\partial y_2} = 2y_2$$
$$F_{21}(y_1, y_2) = \frac{\partial F_2(y_1, y_2)}{\partial y_1} = 2y_1$$
$$F_{22}(y_1, y_2) = \frac{\partial F_2(y_1, y_2)}{\partial y_2} = 1$$

Then: at $\underset{\sim}{y}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underset{\sim}{Y}'(t) = F\,\underset{\sim}{Y}(t), \quad \underset{\sim}{Y}(t) = \underset{\sim}{y}(t) - \underset{\sim}{y}_0 = \underset{\sim}{y}(t)$$

See $p = 2$, $q = 1$, $\Delta = 0$: proper node unstable



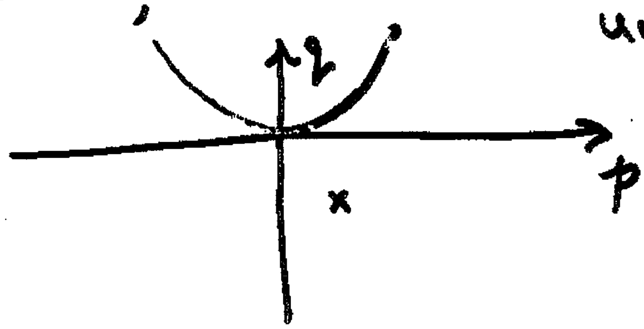
(11.3)

At $\underset{\sim}{y}_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $F = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

$\underset{\sim}{Y}'(t) = F\underset{\sim}{Y}(t)$, $\underset{\sim}{Y}(t) = \underset{\sim}{y}(t) - \underset{\sim}{y}_0$

$$\left[\Rightarrow \begin{aligned} Y_1(t) &= y_1(t) + 1 \\ Y_2(t) &= y_2(t) + 1 \end{aligned} \right]$$

$p = 2$, $q = -3$, $\Delta = 16$: saddle point unstable

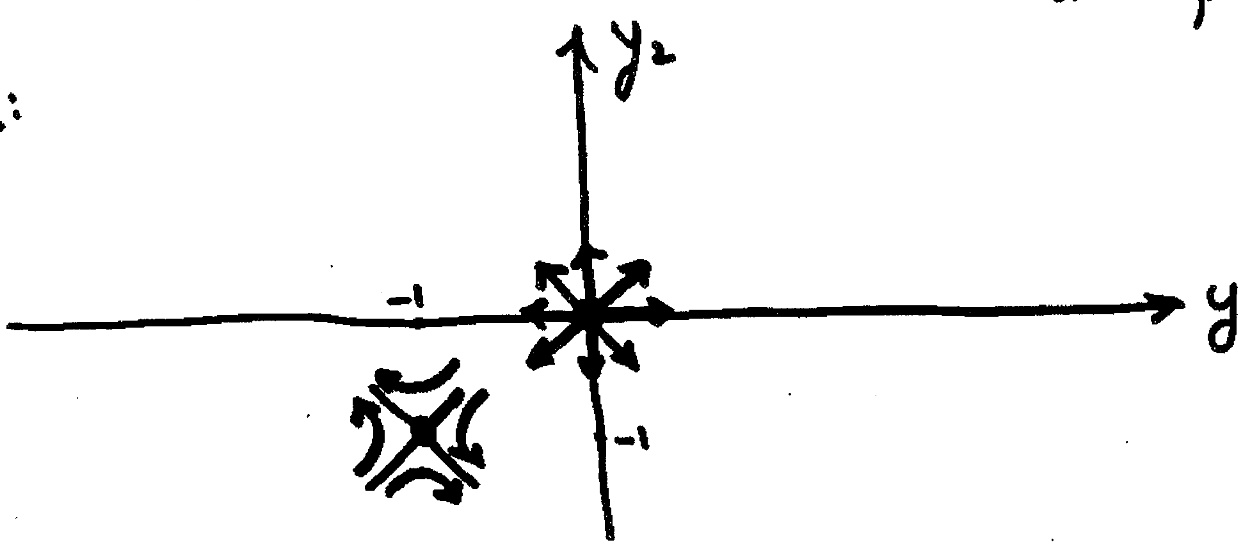


To get directions at saddle, use (say)

$$Y_1'(t) = Y_1(t) - 2Y_2(t)$$

$\Rightarrow$ Y_1 decr. where $Y_1 = 0$, $Y_2 > 0$ (directly above crit. pt.)

So:



Modelling an Arms Race

[L.F. Richardson, "Arms and Insecurity" (c. 1938)
SSH Library U21.7. R48 1978]

Two nations, A and B

$y_1(t) = \begin{cases}\text{war potential} \\ \text{(total armaments)}\end{cases}$ of nation A

$y_2(t) = \begin{cases} \quad " \quad " \end{cases}$ " nation B

at time t.

We suppose there are three main effects influencing $y_1(t)$:

1) $y_1'(t) = k y_2(t) + \cdots$
- models A's reaction to B's armaments

2) $y_1'(t) = -\alpha y_1(t) + \cdots$
- models restraining effect on A of cost of arms

3) $y_1'(t) = g + \cdots$
- models effect on A of grievances towards B

Overall: $y_1'(t) = k\,y_2(t) - \alpha\,y_1(t) + g$

Similarly: $y_2'(t) = l\,y_1(t) - \beta\,y_2(t) + h$

k, l <u>reaction</u> (or <u>defence</u>) coefficients

α, β <u>expense</u> (or <u>fatigue</u>) coefficients

g, h $\begin{cases} \text{grievance terms (if positive)} \\ \text{goodwill terms (if negative)} \end{cases}$

For simplicity, suppose $k, l, \alpha, \beta, g, h$ constant.
(with k, l, α, β, positive)

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} k y_2(t) - \alpha y_1(t) + g \\ l y_1(t) - \beta y_2(t) + h \end{pmatrix} = \begin{pmatrix} F_1(y_1(t), y_2(t)) \\ F_2(y_1(t), y_2(t)) \end{pmatrix}$$

<u>OR</u>

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{bmatrix} -\alpha & k \\ l & -\beta \end{bmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} g \\ h \end{pmatrix} \qquad (11.2)$$

Coupled pair of 1st order <u>linear</u> ODEs
– inhomogeneous – also <u>autonomous</u>

In this case

$$F_{11}(y_1,y_2) = \frac{\partial F_1(y_1,y_2)}{\partial y_1} = -\alpha$$

$$F_{12}(y_1,y_2) = \frac{\partial F_1(y_1,y_2)}{\partial y_2} = k$$

$$F_{21}(y_1,y_2) = \frac{\partial F_2(y_1,y_2)}{\partial y_1} = \ell$$

$$F_{22}(y_1,y_2) = \frac{\partial F_2(y_1,y_2)}{\partial y_2} = -\beta$$

so Jacobian matrix $F = \begin{bmatrix} -\alpha & k \\ \ell & -\beta \end{bmatrix}$ is

independent of point where we evaluate it.

<u>Now</u>: <u>Critical points</u>: $F_1 = 0 = F_2$

$$\Rightarrow \begin{cases} k y_2 - \alpha y_1 + g = 0 \\ \ell y_1 - \beta y_2 + h = 0 \end{cases}$$

<u>OR</u>

$$\begin{bmatrix} -\alpha & k \\ \ell & -\beta \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -\begin{pmatrix} g \\ h \end{pmatrix} \qquad (11.3)$$

Let $q = \det(F) = \alpha\beta - kl$ and suppose $q \neq 0$

Then F has an inverse:

$$F^{-1} = \frac{1}{q}\begin{bmatrix} -\beta & -k \\ -l & -\alpha \end{bmatrix}$$

check $FF^{-1} = F^{-1}F = I$

and solution of (11.3) is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \frac{-1}{q}\begin{bmatrix} -\beta & -k \\ -l & -\alpha \end{bmatrix}\begin{pmatrix} g \\ h \end{pmatrix}$$

$$\left[\Rightarrow \quad y_{10} = \frac{\beta g + kh}{\alpha\beta - kl}, \quad y_{20} = \frac{\alpha h + lg}{\alpha\beta - kl} \right]$$

There is no need to linearize (11.2) – already linear! Just shift origin:

$$Y_1(t) = y_1(t) - y_{10}, \quad Y_2(t) = y_2(t) - y_{20}$$

Then (11.2) becomes

$$\begin{pmatrix} Y_1'(t) \\ Y_2'(t) \end{pmatrix} = F\begin{pmatrix} Y_1(t) + y_{10} \\ Y_2(t) + y_{20} \end{pmatrix} + \begin{pmatrix} g \\ h \end{pmatrix} = F\begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} \quad (11.4)$$

using (11.3): $F\begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = -\begin{pmatrix} g \\ h \end{pmatrix}$

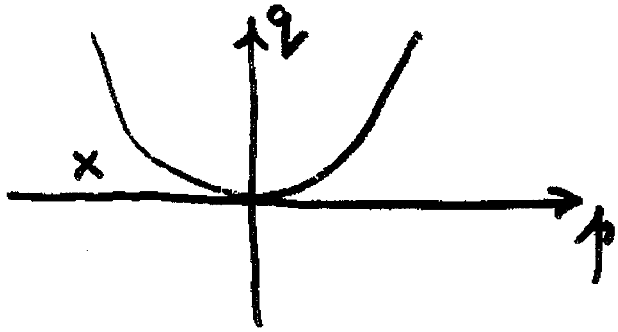
11.8

$$F = \begin{bmatrix} -\alpha & k \\ l & -\beta \end{bmatrix}$$

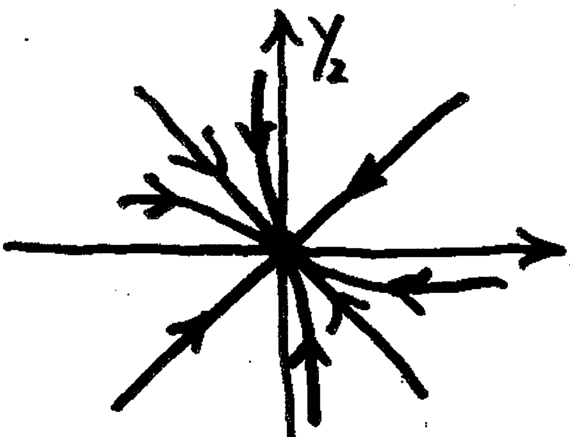
$$p = -(\alpha+\beta) < 0$$
$$q = \alpha\beta - kl$$
$$\Delta = (\alpha+\beta)^2 - 4(\alpha\beta - kl) = (\alpha-\beta)^2 + 4kl > 0$$

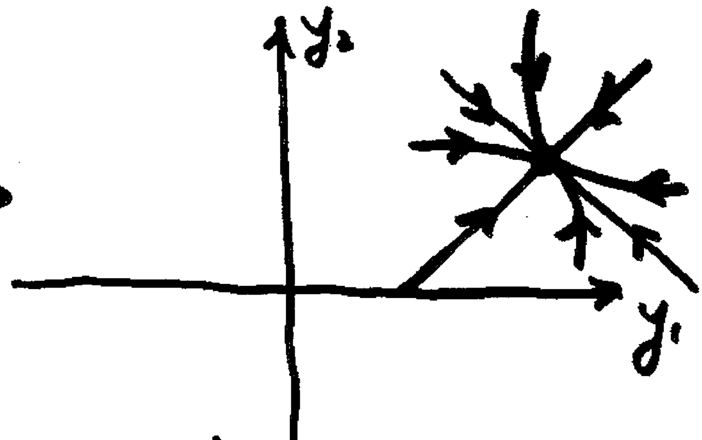
Two cases to consider:

1) $q > 0$ $\quad(\alpha\beta > kl)$

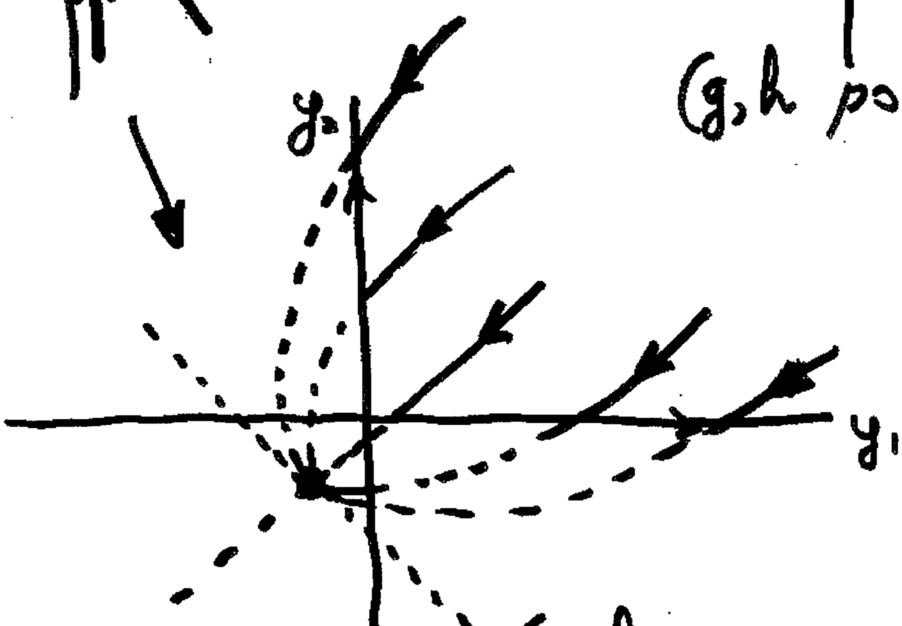


Attractive improper node
- stable and attractive





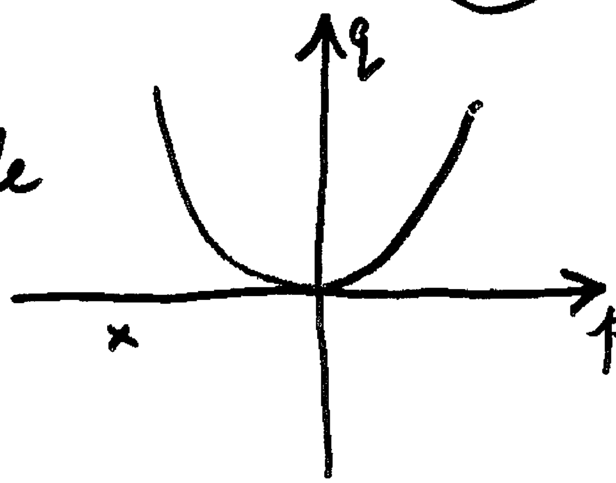
(g, h positive)

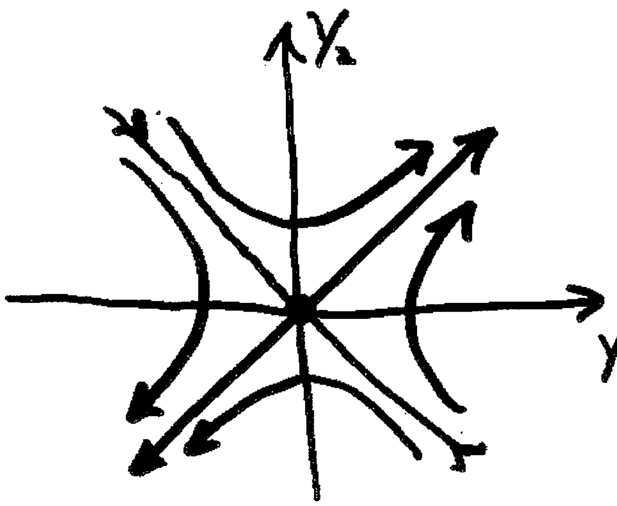


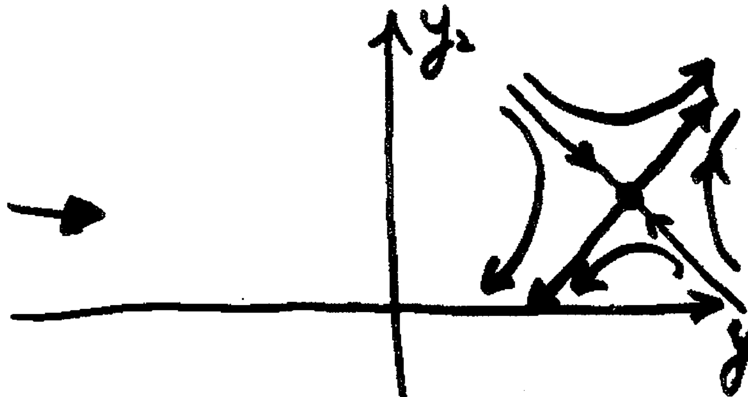
(g, h negative)

2) $q<0$ $(\alpha\beta < kl)$

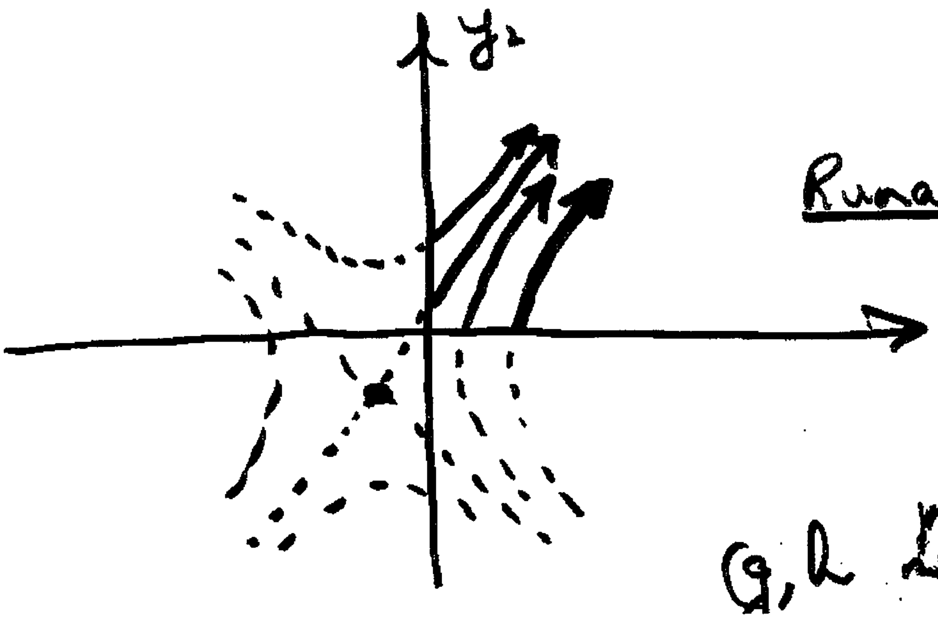
Saddle point - unstable







(g, h ~~positive~~ negative - confused signals)



<u>Runaway Arms Race</u>

(g, h ~~negative~~ positive)

Richardson applied to European Arms Race of 1909–14.

A: Britain – France – Russia

B: Germany – Austro-Hungarian Empire

Suggested $\alpha \approx \beta \approx 0.2$ years^{-1}

$k \approx l \approx 0.9$ years^{-1}

As measures of arms he took total defence budgets. (in M£)

	1909	1910	1911	1912	1913
$y_1 \approx y_2$	100	102	107	119	145

Reasonable fit.

War declared 1914!

Modelling human glucose-insulin system

(M. Braun, DEs and Applications, PS4E QA371.B793 1993)

$g(t)$ = glucose concentration in blood at time t

$h(t)$ = hormone (insulin) concentration ....

Suppose:

$$\left.\begin{aligned} g'(t) &= F_1(g(t), h(t)) \\ h'(t) &= F_2(g(t), h(t)) \end{aligned}\right\} \quad (11.5)$$

autonomous

Suppose $\left.\begin{aligned} F_1(g,h) &= 0 \\ F_2(g,h) &= 0 \end{aligned}\right\} \Rightarrow g = g_0, \; h = h_0$

unique equilibrium point.

Linearize near (g_0, h_0):

$$F_1(g,h) \approx \cancelto{0}{F_1(g_0,h_0)} + F_{11}(g_0,h_0)(g-g_0) + F_{12}(g_0,h_0)(h-h_0)$$

$$F_2(g,h) \approx \cancelto{0}{F_2(g_0,h_0)} + F_{21}(g_0,h_0)(g-g_0) + F_{22}(g_0,h_0)(h-h_0)$$

11.12

Here $F_{12}(g,h) = \dfrac{\partial F_1(g,h)}{\partial h}$ etc.

Then

$$\begin{pmatrix} G'(t) \\ H'(t) \end{pmatrix} = F \begin{pmatrix} G(t) \\ H(t) \end{pmatrix}$$

where

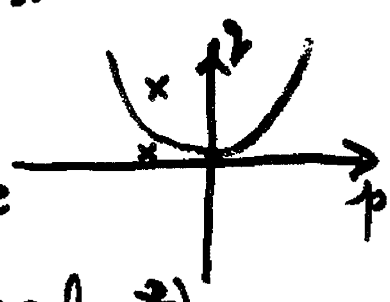
$$F = \begin{bmatrix} F_{11}(g_0,h_0) & F_{12}(g_0,h_0) \\ F_{21}(g_0,h_0) & F_{22}(g_0,h_0) \end{bmatrix}, \quad \begin{array}{l} G(t) = g(t) - g_0 \\ H(t) = h(t) - h_0 \end{array}$$

$$= \begin{bmatrix} -m_1 & -m_2 \\ m_3 & -m_4 \end{bmatrix}, \text{ say.}$$

Typically, m_1, m_2, m_3, m_4 are positive

Then $p = -(m_1 + m_4) < 0$, $\quad q = m_1 m_4 + m_2 m_3 > 0$

$$\Delta = (m_1 + m_4)^2 - 4(m_1 m_4 + m_2 m_3)$$
$$= (m_1 - m_4)^2 - 4 m_2 m_3$$



Get stable + attractive $\underset{\Delta<0}{\text{focus}}$ or $\underset{\Delta\geq 0}{\text{node}}$

Danger sign if q too small ($\lesssim 0.25\ \text{hrs}^{-2}$)

Real problems if $q < 0$

$K \int$ ~~[illegible]~~ 4.5