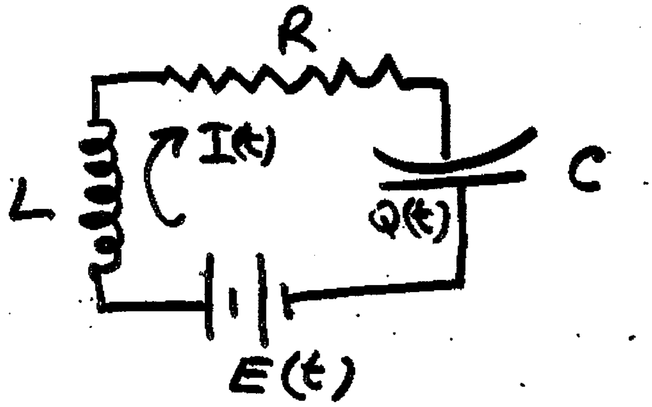
Lec. 12 MATH2100/2010

Sometimes can get path equation directly from "slope equation" $\frac{dy_2}{dy_1} = \cdots$

EX:

(As on pp 1.8, 1.9)



$$\left.\begin{aligned} L\frac{dI(t)}{dt} + RI(t) + \frac{Q(t)}{C} &= E(t) \\ \frac{dQ(t)}{dt} &= I(t) \end{aligned}\right\}$$

Case when $E(t) = 0$:–

$$\begin{pmatrix} Q'(t) \\ I'(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{pmatrix} Q(t) \\ I(t) \end{pmatrix}$$

$\Rightarrow \quad p = -\frac{R}{L} \leq 0, \quad q = \frac{1}{LC} > 0,$

$$\Delta = \frac{R^2}{L^2} - \frac{4}{LC}$$

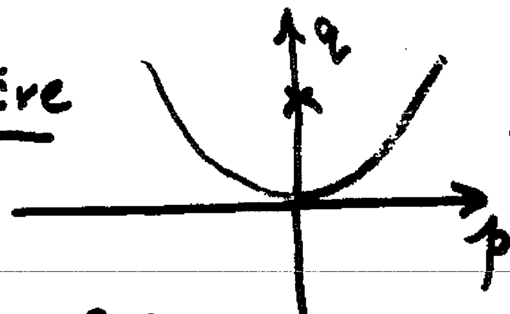
See mathematically is like mass-spring-damper system.

If L, C fixed, R plays role of 'damper.'

Mechanical oscillator $\leftrightarrow$ electrical oscillator

Suppose $R = 0$. Then $p = 0$, $q = \frac{1}{LC} > 0$, $\Delta = \frac{-4}{LC}$

$\Rightarrow$ centre : stable, not attractive



In that case: $\begin{pmatrix} Q'(t) \\ I'(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & 0 \end{bmatrix} \begin{pmatrix} Q(t) \\ I(t) \end{pmatrix}$

$\Rightarrow \quad \frac{dI}{dQ} = \frac{dI/dt}{dQ/dt} = \frac{-\frac{1}{LC} Q}{I}$ "Slope equation"

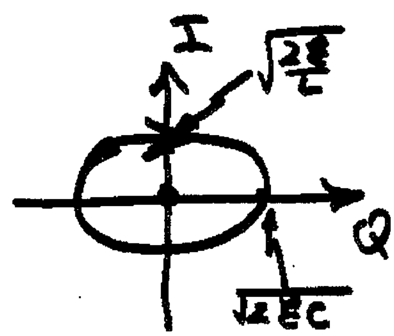
"path equation"

$\Rightarrow \quad I\,dI = -\frac{1}{LC} Q\,dQ$

$\Rightarrow \quad \frac{1}{2} I^2 = -\frac{1}{LC} \frac{1}{2} Q^2 + D$

OR $\quad \frac{1}{2} \frac{Q^2}{C} + \frac{1}{2} L I^2 = \mathcal{E}$ (const.)

– ellipse in QI-plane



Modelling an epidemic

Population of n individuals ($\simeq$ constant).

$x(t) =$ no. susceptible at time t

$y(t) =$ no. infectious at time t

$z(t) =$ no. immune at time t

$$x(t) + y(t) + z(t) = n$$

Suppose x, y, z large — smoothly-varying functions of t. (Only non-negative values are meaningful.)

Model:

$$\left.\begin{aligned} x'(t) &= -\beta x(t) y(t) \\ y'(t) &= \beta x(t) y(t) - \gamma y(t) \\ z'(t) &= \gamma y(t) \end{aligned}\right\} \qquad (12.1)$$

β, γ positive constants

Really only two dependent variables, as

$$z(t) = n - x(t) - y(t)$$

[<u>Check</u>: $z'(t) = 0 - x'(t) - y'(t)$ ✓]

So:

$$\left.\begin{aligned} x'(t) &= -\beta x(t)\,y(t) &&= F_1(x(t), y(t)) \\ y'(t) &= \beta x(t) y(t) - \gamma y(t) &&= F_2(x(t), y(t)) \end{aligned}\right\} \quad (12.$$

<u>Critical points:</u> $F_1(x, y) = 0 = F_2(x, y)$

$$\Rightarrow \quad -\beta x y = 0 \text{ and } \beta x y - \gamma y = 0$$

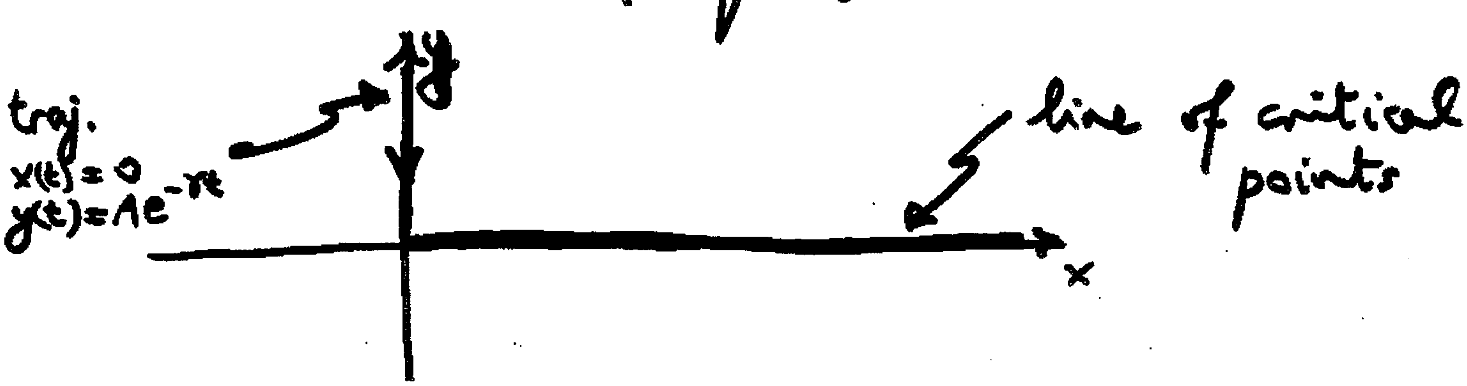
Add: $\Rightarrow \; -\gamma y = 0 \;\Rightarrow\; y = 0$

Then x is arbitrary.

So: $\begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ is a critical point for

<u>every</u> x_0.

[<u>Meaning</u>: No infections, any no. of susceptibles] $\Rightarrow$ equilibrium.



Linearization:

$$F_{11}(x,y) = \frac{\partial F_1(x,y)}{\partial x} = -\beta y$$

$$F_{12}(x,y) = \frac{\partial F_1(x,y)}{\partial y} = -\beta x$$

$$F_{21}(x,y) = \frac{\partial F_2(x,y)}{\partial x} = \beta y$$

$$F_{22}(x,y) = \frac{\partial F_2(x,y)}{\partial y} = \beta x - \gamma$$

Then, at $(x_0, 0)$,

$$F = \begin{bmatrix} 0 & -\beta x_0 \\ 0 & \beta x_0 - \gamma \end{bmatrix}$$

See $q = 0$: doesn't belong to Types 1 – 6

Linearized system is

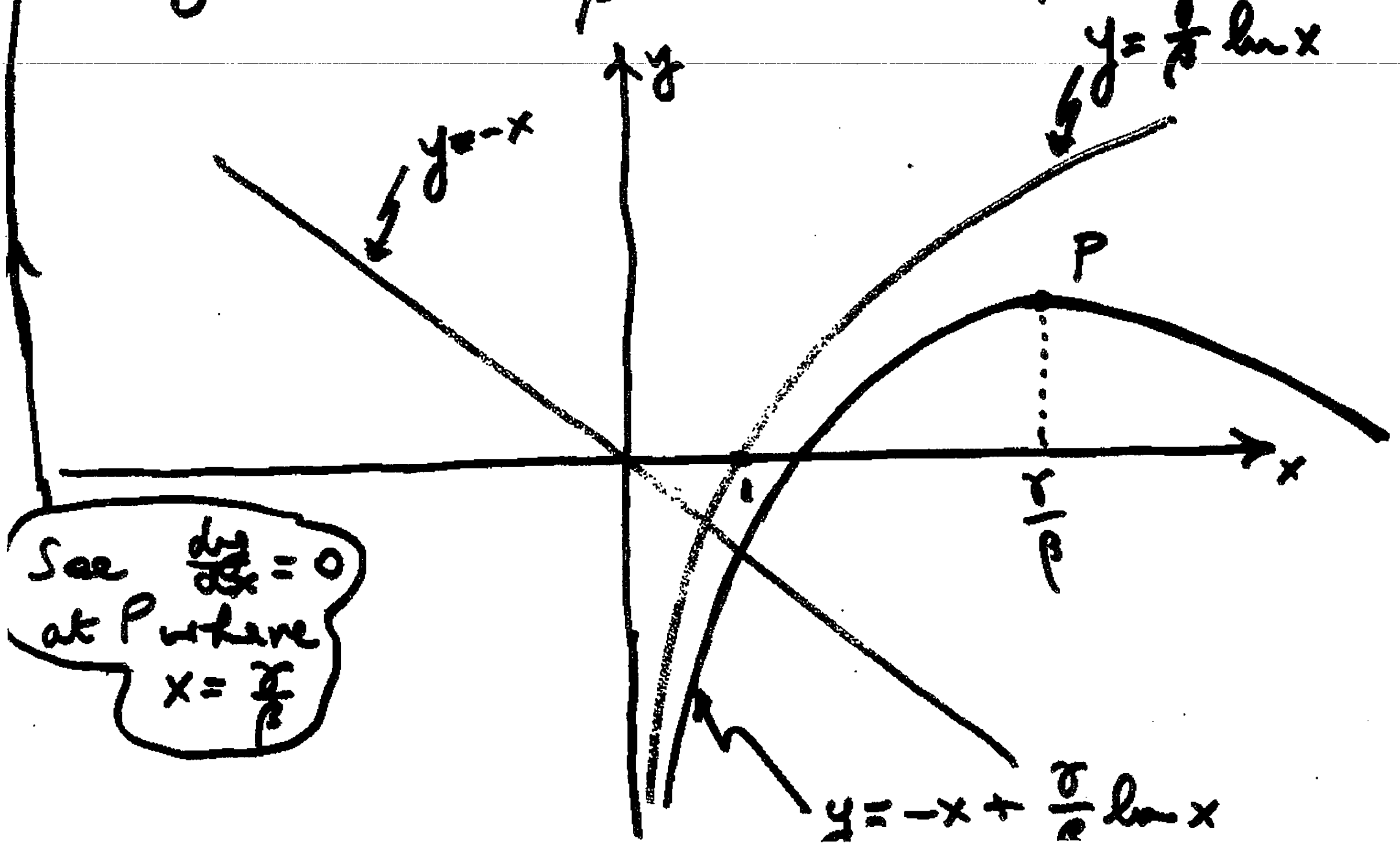
$$\begin{pmatrix} X'(t) \\ Y'(t) \end{pmatrix} = F \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \qquad \begin{aligned} X(t) &= x(t) - x_0 \\ Y(t) &= y(t) \end{aligned}$$

It is easy to solve the linearized equations (try it!), but in this case better to find path equation from slope equation for original system (12.2):–

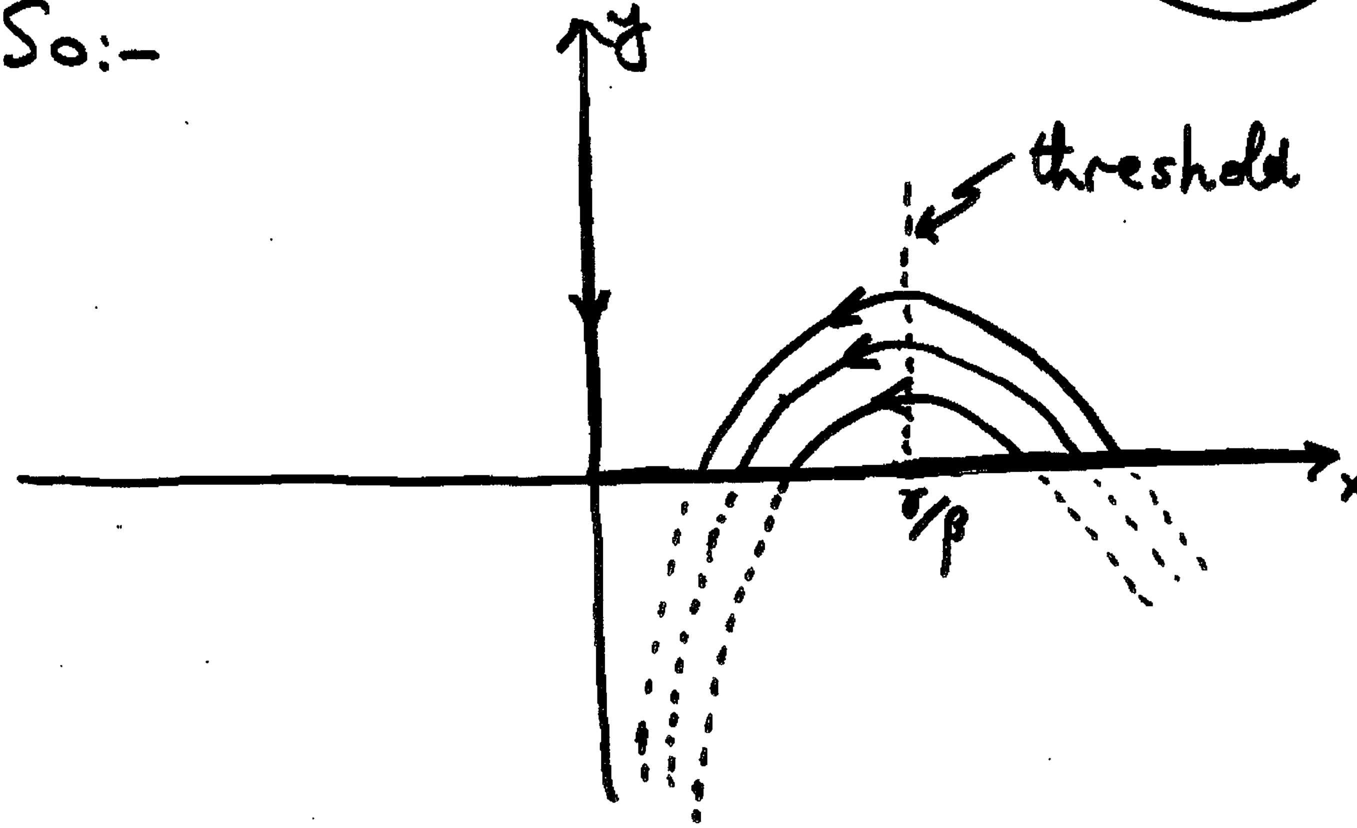
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\beta x y - \gamma y}{-\beta x y} = \frac{\beta x - \gamma}{-\beta x}$$

i.e. $\dfrac{dy}{dx} = -1 + \dfrac{\gamma}{\beta x}$ $\quad (x \neq 0,\ y \neq 0)$ — slope equation

$\Rightarrow y(x) = -x + \dfrac{\gamma}{\beta} \ln x + C$ – path equation



So:-



To get arrows: use $\frac{dx}{dt} = -\beta x y$

$\Rightarrow$ x decreasing for $x>0$, $y>0$.

See if $x > \frac{\gamma}{\beta}$, a small perturbation (= intro. of a small no. of infectious) leads to epidemic, which peaks, then dies out.

If $x < \frac{\gamma}{\beta}$, small perturbation does not lead to epidemic.

So $x = \frac{\gamma}{\beta}$ is a threshold value of susceptibles.

<u>Note</u>: None of critical points $\binom{x_0}{0}$ is stable and attractive

TOPIC 2 : Laplace Transforms

[K : Chapter 5]

Given one function $f(t)$, $t \geq 0$,
we define another function

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

[if integral exists !!]

We call $F(s)$ the Laplace Transform
of $f(t)$, and we write

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

Ex1: $f(t) = K$ (constant)

Then $$F(s) = \int_0^\infty e^{-st} K\, dt$$

$$= \left[\frac{e^{-st} K}{-s}\right]_{t=0}^{t=\infty}$$

$$= \frac{K}{s}$$

(assuming that $e^{-st} \to 0$ as $t \to \infty$.
ok if $s > 0$
(or even s complex, with $Re(s) > 0$)

So: $$\boxed{\mathcal{L}(K) = \frac{K}{s}}$$

Ex2: $f(t) = t$

Then
$$\mathcal{L}(t) = F(s) = \int_0^\infty \underbrace{t}_{v}\, \underbrace{e^{-st}}_{u'}\, dt$$

Integrate by parts:

$$F(s) = \Big[\underbrace{\frac{e^{-st}}{-s}}_{u}\underbrace{t}_{v}\Big]_{t=0}^{t=\infty} + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= 0 + \frac{1}{s}\mathcal{L}(1)$$

$$= \frac{1}{s^2}$$

So: $\boxed{\mathcal{L}(t) = \frac{1}{s^2}}$

EX3: $\mathcal{L}(t^n) = \int_0^\infty \underbrace{e^{-st}}_{v'}\underbrace{t^n}_{u}\,dt$ ($n > 0$, integer)

$$= \Big[\underbrace{\frac{e^{-st}}{-s}}_{v}\underbrace{t^n}_{u}\Big]_{t=0}^{t=\infty} + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt$$

$$= 0 + \frac{n}{s}\mathcal{L}(t^{n-1})$$

$$= \frac{n(n-1)}{s^2}\mathcal{L}(t^{n-2}) \text{ etc.}$$

So, by induction:- $\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$

[n positive integer. See OK also for $n=0$]

EX 4: $f(t) = e^{at}$ a = constant

$$F(s) = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt$$

$$= \left[\frac{e^{(a-s)t}}{a-s} \right]_{t=0}^{t=\infty}$$

$$= \frac{-1}{a-s}$$ (provided $s > a$ – or at least $Re(s) > Re(a)$)

So: $\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}}$

Important that this is true even for $a = b + ic$ complex – provided $s > b$ – or even s complex with $Re(s) > b$

Summary:

1) Sometimes easy to get path equation for trajectories by integrating the slope equation $\frac{dy_2}{dy_1} = \ldots$

2) LRC-circuit is electrical analogue of mass-spring-damper system . . .

3) Systems with $q = \det(F) = 0$ at a critical point can be interesting (e.g. in epidemic model).

4) Understand definition of Laplace Transform and learn $\mathcal{L}(t^n)$, $\mathcal{L}(e^{at})$

K § 4.5 esp. pp 156   § 6.1