Lec. 13 MATH2100/2010

Laplace Transform of a function $f(t)$ $(t \geq 0)$ is defined to be:

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

We saw:

$$\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, \dots \quad s > 0} \tag{13.1}$$

and

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > Re(a)} \tag{13.2}$$

Next note the general result that, if $f(t) = A g(t) + B h(t)$, then

$$\boxed{\mathcal{L}(f(t)) = A\mathcal{L}(g(t)) + B\mathcal{L}(h(t))} \tag{13.3}$$

OR

$$\boxed{F(s) = A\, G(s) + B\, H(s)}$$

(13.2)

Pf: $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\infty e^{-st}\left[A g(t) + B h(t)\right] dt$$

$$= A\int_0^\infty e^{-st} g(t)\,dt + B\int_0^\infty e^{-st} h(t)\,dt$$

$$= A\,\mathcal{L}(g(t)) + B\,\mathcal{L}(h(t))$$

This result (13.3) says that taking (or forming) a Laplace Transform is a <u>linear operation</u>.

EX: $f(t) = 3t^4 + 6t + 8e^{-2t}$

$$\Rightarrow \quad F(s) = \frac{3.4!}{s^5} + \frac{6.1!}{s^2} + \frac{8}{s+2}$$

$$= \frac{72}{s^5} + \frac{6}{s^2} + \frac{8}{s+2}$$

EX: $f(t) = e^{i\alpha t}$, α real

$\Rightarrow$

$$F(s) = \frac{1}{s - i\alpha}$$

$$= \frac{s + i\alpha}{(s-i\alpha)(s+i\alpha)}$$

$$= \frac{s+i\alpha}{s^2+\alpha^2}$$

$$= \frac{s}{s^2+\alpha^2} + i\,\frac{\alpha}{s^2+\alpha^2}$$

So

$$\mathcal{L}(e^{i\alpha t}) = \frac{s}{s^2+\alpha^2} + i\,\frac{\alpha}{s^2+\alpha^2} \qquad (13.4)$$

But $\quad e^{i\alpha t} = \cos(\alpha t) + i\sin(\alpha t)$

So

$$\mathcal{L}(e^{i\alpha t}) = \mathcal{L}(\cos(\alpha t)) + i\,\mathcal{L}(\sin(\alpha t)) \qquad (13.5)$$

Comparing (13.4) and (13.5) (equating real and imaginary parts) we get

$$\boxed{\mathcal{L}(\cos(\alpha t)) = \frac{s}{s^2+\alpha^2}} \qquad (13.6)$$

$$\boxed{\mathcal{L}(\sin(\alpha t)) = \frac{\alpha}{s^2+\alpha^2}} \qquad (13.7)$$

Could get (13.6), (13.7) without using complex numbers, but not as easy :—

13.4

$$\mathcal{L}(\cos(\alpha t)) = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{\cos(\alpha t)}_{v'}\, dt$$

$$= \Big[\underbrace{e^{-st}}_{u} \underbrace{\frac{\sin(\alpha t)}{\alpha}}_{v}\Big]_{t=0}^{t=\infty} - \int_0^\infty \underbrace{(-s e^{-st})}_{u'} \underbrace{\frac{\sin(\alpha t)}{\alpha}}_{v}\, dt$$

$$= \frac{s}{\alpha} \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{\sin(\alpha t)}_{v'}\, dt$$

$$= \frac{s}{\alpha}\Big\{\Big[\underbrace{e^{-st}}_{u} \underbrace{\frac{-\cos(\alpha t)}{\alpha}}_{v}\Big]_{t=0}^{t=\infty}$$

$$- \int_0^\infty \underbrace{(-s e^{-st})}_{u'} \underbrace{\frac{-\cos(\alpha t)}{\alpha}}_{v}\, dt\Big\}$$

$$= \frac{s}{\alpha}\Big\{\Big[0 - \Big(\frac{-1}{\alpha}\Big)\Big] - \frac{s}{\alpha}\int_0^\infty e^{-st}\cos(\alpha t)\, dt\Big\}$$

$$= \frac{s}{\alpha^2} - \frac{s^2}{\alpha^2}\,\mathcal{L}(\cos(\alpha t))$$

So

$$\Big(1 + \frac{s^2}{\alpha^2}\Big)\mathcal{L}(\cos(\alpha t)) = \frac{s}{\alpha^2}$$

$$\Rightarrow \mathcal{L}(\cos(\alpha t)) = \frac{s}{s^2+\alpha^2}$$

as before.

Suppose we are given $f(t)$, and so find

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Then $$\mathcal{L}(e^{-at} f(t)) = \int_0^\infty e^{-st} e^{-at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

i.e. $$\boxed{\mathcal{L}(e^{-at} f(t)) = F(s+a)} \qquad (13.8)$$

First Shifting Theorem
(Shifting in s)

EX: $\mathcal{L}(1) = \frac{1}{s}$

$\Rightarrow \mathcal{L}(e^{-at}) = \frac{1}{s+a}$

— as before in (13.2)

But now we can get new results:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0$$

$$\Rightarrow \boxed{\mathcal{L}(t^n e^{-at}) = \frac{n!}{(s+a)^{n+1}} \quad \begin{array}{l} n = 0, 1, 2, \ldots \\ (s+a) > 0 \end{array}} \qquad (13.9)$$

$$\mathcal{L}(\cos(\alpha t)) = \frac{s}{s^2+\alpha^2} \qquad (s>0)$$

$$\Rightarrow \boxed{\mathcal{L}(\cos(\alpha t) e^{-at}) = \frac{s+a}{(s+a)^2+\alpha^2} \quad (s+a) > 0} \qquad (13.10)$$

$$\mathcal{L}(\sin(\alpha t)) = \frac{\alpha}{s^2+\alpha^2} \qquad (s>0)$$

$$\Rightarrow \boxed{\mathcal{L}(\sin(\alpha t) e^{-at}) = \frac{\alpha}{(s+a)^2+\alpha^2} \quad (s+a) > 0} \qquad (13.11)$$

We saw $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$, $n = 0, 1, 2, \dots$
$s > 0$

Consider more generally

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a \, dt$$

$(a > 0)$

Change variable of integration:

$t \rightarrow x = st$ so $dx = s\,dt$

$t = 0 \Leftrightarrow x = 0$ $\qquad$ $t = \infty \Leftrightarrow x = \infty$

We get:-

$$\mathcal{L}(t^a) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \frac{dx}{s}$$

$$= \frac{1}{s^{a+1}} \underbrace{\int_0^\infty e^{-x} x^a \, dx}_{\Gamma(a+1)}$$

$$\boxed{\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}} \qquad (13.12)$$

13.8

Here

$$\boxed{\Gamma(p) \overset{\text{def.}}{=} \int_0^\infty e^{-x} x^{p-1}\, dx \quad , \quad p \geq 1 \qquad (13.13)}$$

is the Gamma function

The Gamma function generalizes the factorial.

See

$$\boxed{\Gamma(n+1) = \int_0^\infty \underbrace{e^{-x}}_{u'} \underbrace{x^n}_{v} dx = n!} \qquad (13.14)$$

$$\Big[\ \Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1$$

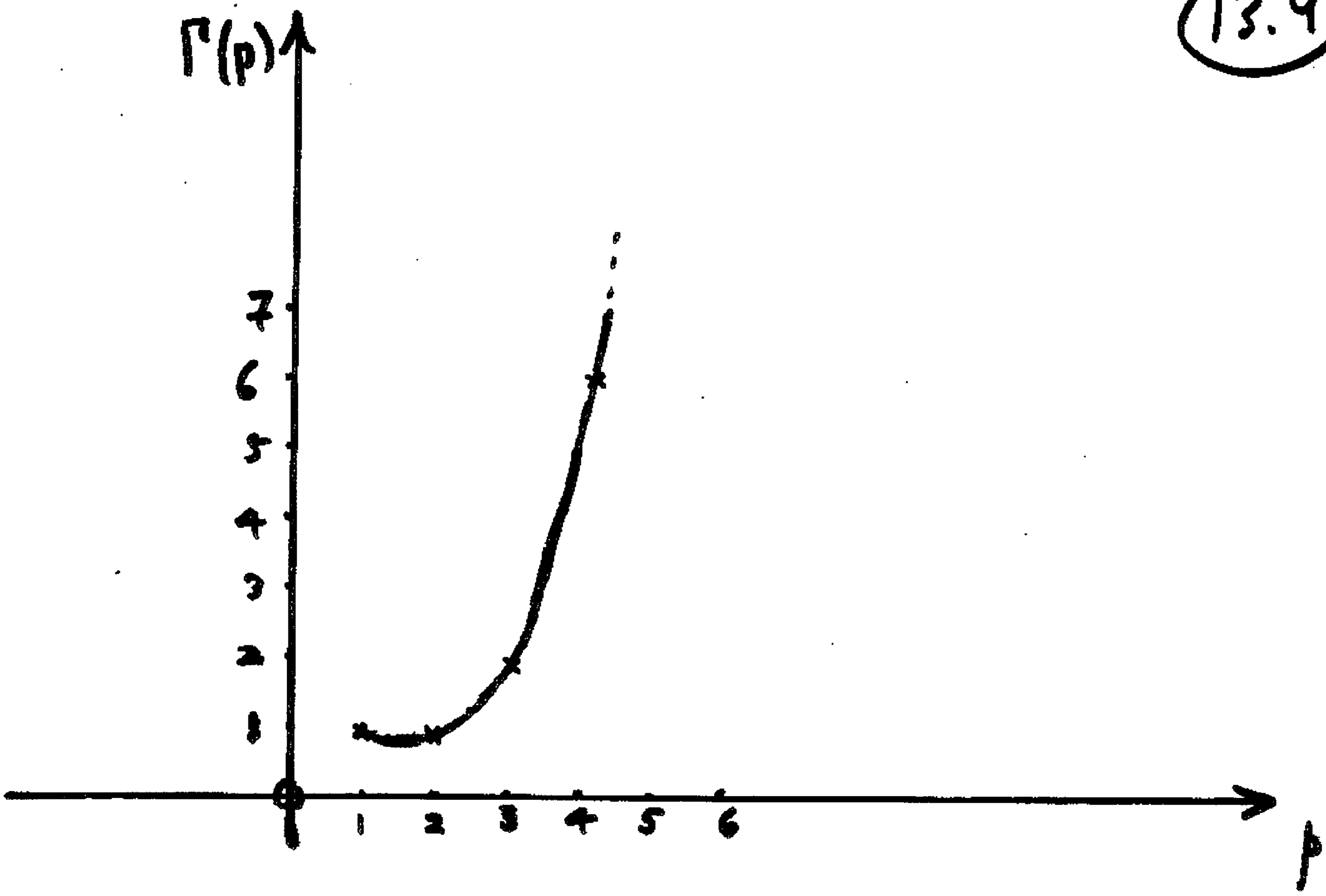
$$\Gamma(n+1) = \Big[\cancel{-e^{-x}\underbrace{x^n}_{v}}\Big]_0^\infty + n\int_0^\infty e^{-x} x^{n-1} dx \ , \quad n=1,2,\dots$$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n)\ , \quad n=1,2,\dots} \qquad (13.15)$$

$$\Gamma(2) = 1\,\Gamma(1) = 1.1$$

$$\Gamma(3) = 2\,\Gamma(2) = 2.1.1$$

$$\Gamma(4) = 3\,\Gamma(3) = 3.2.1.1 \ \Big]$$



What happens for $p < 1$?

<u>Note</u>: On MAPLE, called GAMMA

Consider again the definition:-

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

<u>Question</u>: When is $\mathcal{L}(f(t))$ <u>well-defined</u>?

Obv. doesn't work for, say,

$$f(t) = e^{t^2}$$

<u>Answer:</u> In general, we need

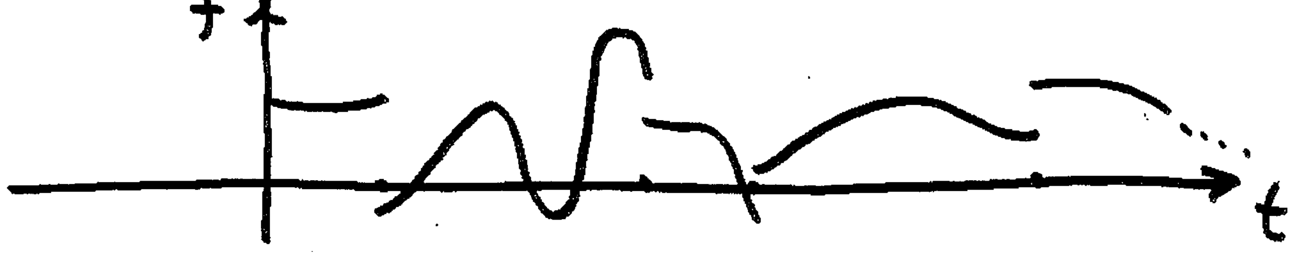
1) $|f(t)| \leq Me^{\gamma t}$ for <u>all</u> $t > 0$
for <u>some</u> $M > 0$
for <u>some</u> $\gamma > 0$

2) $f(t)$ should be continuous, or at least <u>piecewise continuous</u>, for $0 \leq t < \infty$

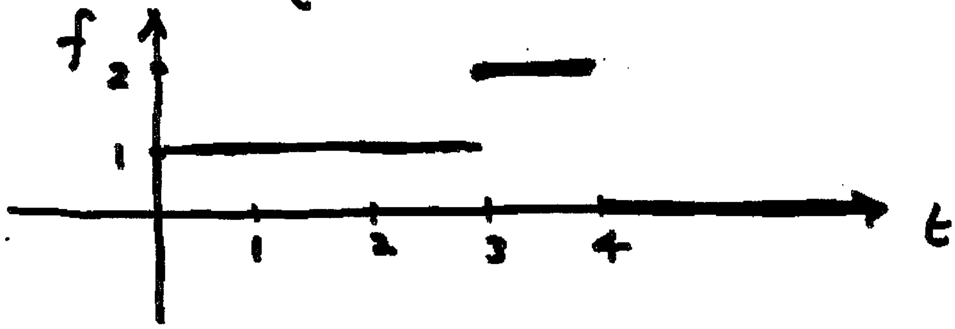
When 1) and 2) hold, there are no problems with the integral $\int_0^\infty e^{-st} f(t)\,dt$, and

$F(s)$ is defined, for $s > \gamma$.

<u>Piecewise continuous:</u> f made up of a finite number of continuous pieces on every subinterval $[0,T]$, and limits from R and L are finite at every point of discontinuity:



EX: $f(t) = \begin{cases} 1 & 0 \leq t < 3 \\ 2 & 3 \leq t < 4 \\ 0 & t \geq 4 \end{cases}$



See 1) and 2) Rolld.

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^3 e^{-st} \cdot 1\,dt + \int_3^4 e^{-st} \cdot 2\,dt + 0$$

$$= \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=3} + \left[\frac{2e^{-st}}{-s}\right]_{t=3}^{t=4}$$

$$= \frac{e^{-3s}-1}{-s} + \frac{2e^{-4s}-2e^{-3s}}{-s}$$

$$= \frac{1+e^{-3s}-2e^{-4s}}{s}$$

<u>Note</u>: If we changed values of f at $t = 3, 4$ that wouldn't change F

Summary:

1) Taking Laplace Transform is a <u>linear operation</u>.

2) Be able to find $\mathcal{L}(\cos(\alpha t))$, $\mathcal{L}(\sin(\alpha t))$

3) Understand and be able to use <u>First Shifting Theorem</u>

4) Understand definition of Gamma function.

5) Understand conditions under which $\mathcal{L}(f(t))$ is well-defined.

6) Understand idea of a <u>piece-wise continuous</u> function.

7) No need to learn Laplace Transforms by heart (except perhaps for K, t^n, $e^{\alpha t}$, $\cos(\alpha t)$ and $\sin(\alpha t)$), but be able to derive them from definition.

K §5.1