Lec. 16 MATH2100/2010

Consider an initial value problem (I.V.P.) for a second-order ODE:

$$\left.\begin{array}{l} y''(t) - 4y'(t) + 4y(t) = 5te^{2t} + 7 \\ y(0) = 2, \quad y'(0) = -1 \end{array}\right\} \qquad (16.1)$$

Taking Laplace Transform of each side of ODE, we get

$$\mathcal{L}(y''(t)) - 4\mathcal{L}(y'(t)) + 4\mathcal{L}(y(t)) = 5\mathcal{L}(te^{2t} + 7)$$

Setting $\mathcal{L}(y(t)) = Y(s)$, we then have

$$[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 4Y(s) = \frac{5}{(s-2)^2} + \frac{7}{s}$$

and so

$$\left[s^2 Y(s) - 2s + 1\right] - 4\left[sY(s) - 2\right] + 4Y(s) = \frac{5}{(s-2)^2} + \frac{7}{s}$$

$\Rightarrow$

$$\underbrace{(s^2 - 4s + 4)}_{(s-2)^2} Y(s) = 2s - 9 + \frac{5}{(s-2)^2} + \frac{7}{s}$$

$$\Rightarrow \quad Y(s) = \underbrace{\frac{2s-9}{(s-2)^2}}_{= \frac{2(s-2)-5}{(s-2)^2}} + \frac{5}{(s-2)^4} + \underbrace{\frac{7}{s(s-2)^2}}_{= \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}}$$

$$\Downarrow$$

$$A = \frac{7}{4}, \; B = -\frac{7}{4}, \; C = \frac{7}{2}$$

$$= \frac{2}{s-2} - \frac{5}{(s-2)^2} + \frac{5}{(s-2)^4} + \frac{7}{4}\frac{1}{s} - \frac{7}{4}\frac{1}{s-2} + \frac{7}{2}\frac{1}{(s-2)^2}$$

$$= \frac{1}{4}\frac{1}{s-2} - \frac{3}{2}\frac{1}{(s-2)^2} + \frac{5}{(s-2)^4} + \frac{7}{4}\frac{1}{s}$$

Then

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{1}{4}e^{2t} - \frac{3}{2}te^{2t} + \frac{5}{6}t^3 e^{2t} + \frac{7}{4}$$

16.3

General I.V.P. for 2nd order, linear ODE with constant coefficients:—

$$\left.\begin{aligned} y''(t) + a y'(t) + b y(t) &= r(t) \\ y(0) = K_0 \ , \quad y'(0) &= K_1 \end{aligned}\right\} \qquad (16.2)$$

Take Laplace Transform of both sides of ODE:-

$$[s^2 Y(s) - s\,y(0) - y'(0)] + a\,[sY(s) - y(0)] + b\,Y(s) = R(s)$$

where

$$\mathcal{L}(y(t)) = Y(s) \qquad \mathcal{L}(r(t)) = R(s)$$

$$\Rightarrow \quad (s^2 + as + b)\,Y(s) = (s+a)\,y(0) + y'(0) + R(s)$$

$$\Rightarrow \quad Y(s) = \frac{(s+a)K_0 + K_1 + R(s)}{(s^2 + as + b)}$$

<u>Next</u>: Try to bring RHS to a form where can recognize it as the Laplace Transform of a known function (with the help of tables).

<u>Some jargon</u>:-

$$Q(s) = \frac{1}{s^2 + as + b}$$

is called the <u>transfer function</u>. (Does not depend on form of $\left\{\begin{array}{l}\text{forcing term}\\ \text{input function}\end{array}\right.$ $r(t)$ nor on I.C.s.)

So: $\quad Y(s) = \left[(s+a)K_0 + K_1 + R(s)\right] Q(s)$

Note the (re-)appearance of the <u>characteristic quadratic</u> as the denominator of $Q(s)$.

<u>Special Case</u>: $K_0 = 0 = K_1$

$\Rightarrow \quad Y(s) = R(s)\,Q(s)$

$\mathcal{L}(\text{output}) = \mathcal{L}(\text{input}) \times$ transfer function

($y(t)$ = output, $r(t)$ = input)

We can use this approach also for linear systems of ODEs, even inhomogeneous:

$$\left.\begin{aligned} \underset{\sim}{y}'(t) &= A\,\underset{\sim}{y}(t) + \underset{\sim}{g}(t) \\ \underset{\sim}{y}(0) &= \underset{\sim}{y}_0 \end{aligned}\right\} \qquad (16.3)$$

$$\Rightarrow \quad s\,\underset{\sim}{Y}(s) - \underset{\sim}{y}_0 = A\,\underset{\sim}{Y}(s) + \underset{\sim}{G}(s)$$

$$\Rightarrow \quad (A - sI)\,\underset{\sim}{Y}(s) = -\underset{\sim}{y}_0 - \underset{\sim}{G}(s)$$

$$\Rightarrow \quad \underset{\sim}{Y}(s) = -(A - sI)^{-1}\left(\underset{\sim}{y}_0 + \underset{\sim}{G}(s)\right)$$

Here

$$\underset{\sim}{Y}(s) = \mathcal{L}(\underset{\sim}{y}(t)) = \begin{pmatrix} \mathcal{L}(y_1(t)) \\ \mathcal{L}(y_2(t)) \\ \vdots \\ \mathcal{L}(y_n(t)) \end{pmatrix}$$

$$\underset{\sim}{G}(s) = \mathcal{L}(\underset{\sim}{g}(t)) = \begin{pmatrix} \mathcal{L}(g_1(t)) \\ \mathcal{L}(g_2(t)) \\ \vdots \\ \mathcal{L}(g_n(t)) \end{pmatrix}$$

Ex: $$\left.\begin{aligned} \underset{\sim}{y}'(t) &= A\,\underset{\sim}{y}(t) + \underset{\sim}{g}(t) \\ \underset{\sim}{y}(0) &= \underset{\sim}{y}_0 \end{aligned}\right\} \qquad (16.3)$$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}, \quad \underset{\sim}{g}(t) = \begin{pmatrix} -6e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

$$\underset{\sim}{y}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For system of 2 coupled equations, easiest to write out components:–

$$y_1'(t) = -3y_1(t) + y_2(t) - 6e^{-2t}$$

$$y_2'(t) = y_1(t) - 3y_2(t) + 2e^{-2t}$$

Take Laplace Transforms, setting

$$\mathcal{L}(y_1(t)) = Y_1(s), \quad \mathcal{L}(y_2(t)) = Y_2(s) \; :–$$

$$\Rightarrow \left.\begin{aligned} [sY_1(s) - \underbrace{y_1(0)}_{=1}] &= -3Y_1(s) + Y_2(s) - \frac{6}{s+2} \\ [sY_2(s) - \underbrace{y_2(0)}_{=0}] &= Y_1(s) - 3Y_2(s) + \frac{2}{s+2} \end{aligned}\right\} \qquad (16.4)$$

$$\Rightarrow \quad \left. \begin{aligned} (s+3)\,Y_1(s) - Y_2(s) &= 1 - \frac{6}{s+2} = \frac{s-4}{s+2} \\ (s+3)\,Y_2(s) - Y_1(s) &= \frac{2}{s+2} \end{aligned} \right\} \qquad (16.5)$$

We solve this pair of coupled algebraic equations for $Y_1(s)$ and $Y_2(s)$, then invert.

So:

Multiply second equation by $(s+3)$, add to first:-

$$\underbrace{\left[(s+3)^2 - 1\right]}_{\substack{s^2+6s+8 \\ \parallel \\ (s+2)(s+4)}} Y_2(s) + 0.Y_1(s) = \frac{2(s+3)}{(s+2)} + \frac{s-4}{s+2} = \frac{3s+2}{s+2}$$

$$\Rightarrow \quad Y_2(s) = \frac{(3s+2)}{(s+2)^2\,(s+4)} \qquad (16.6a)$$

Then second of equations (16.5) gives:

$$\begin{aligned} Y_1(s) &= (s+3)\,Y_2(s) - \frac{2}{s+2} \\ &= \frac{(3s+2)(s+3)}{(s+2)^2(s+4)} - \frac{2}{s+2} \end{aligned}$$

$$= \frac{3s^2 + 11s + 6 - 2(s^2+6s+8)}{(s+2)^2(s+4)}$$

$$Y_1(s) = \frac{s^2 - s - 10}{(s+2)^2(s+4)} \qquad (16.6b)$$

Now invert (16.6a,b) to get solution:–

$$Y_1(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+4}$$

$$\Rightarrow \quad s^2 - s - 10 = A(s+2)(s+4) + B(s+4) + C(s+2)^2$$

$s = -2 \Rightarrow \quad 4+2-10 = 0 + 2B + 0 \quad \Rightarrow \quad B = -2$

$s = -4 \Rightarrow \quad 16+4-10 = 0 + 0 + 4C \quad \Rightarrow \quad C = \frac{5}{2}$

Coeff. of s^2: $\quad 1 = A + 0 + C \quad \Rightarrow A = -3/2$

So:– $\quad Y_1(s) = \frac{-3/2}{(s+2)} - \frac{2}{(s+2)^2} + \frac{5/2}{(s+4)}$

$$\Rightarrow y_1(t) = -\frac{3}{2}e^{-2t} - 2te^{-2t} + \frac{5}{2}e^{-4t}$$

Similarly (check!):–

$$y_2(t) = -2te^{-2t} + \frac{5}{2}e^{-2t} - \frac{5}{2}e^{-4t}$$

EX: (~~[illegible]~~)

$$\left.\begin{aligned}\underset{\sim}{y}'(t) &= A\,\underset{\sim}{y}(t) + \underset{\sim}{g}(t)\\ \underset{\sim}{y}(0) &= \underset{\sim}{y}_0\end{aligned}\right\} \qquad (16.3)$$

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad \underset{\sim}{g}(t) = \begin{pmatrix} 64t\,u(t-1) \\ 0 \end{pmatrix}, \quad \underset{\sim}{y}_0 = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

↑ forcing (input) comes on at $t=1$

In component form:–

$$y_1'(t) = 2y_1(t) + 4y_2(t) + 64t\,u(t-1)$$

$$\left.\begin{aligned}\Rightarrow \quad y_1'(t) &= 2y_1(t) + 4y_2(t) + 64(t-1)\,u(t-1) \\ &\qquad + 64\,u(t-1) \\ y_2'(t) &= y_1(t) + 2y_2(t)\end{aligned}\right\}$$

Now take Laplace Transforms of each equation, setting $\mathcal{L}(y_1(t)) = Y_1(s)$,

$$\mathcal{L}(y_2(t)) = Y_2(s)$$

$\Rightarrow$ (16.10)

$$\Big[sY_1(s) - \underbrace{y_1(0)}_{\overset{\shortparallel}{-4}}\Big] = 2Y_1(s) + 4Y_2(s) + e^{-s}\cdot\frac{64}{s^2} + e^{-s}\frac{64}{s}$$

(Second Shifting Theorem !)

$$\Big[sY_2(s) - \underbrace{y_2(0)}_{\overset{\shortparallel}{-4}}\Big] = Y_1(s) + 2Y_2(s)$$

$$\Rightarrow \quad \left.\begin{aligned} (s-2)Y_1(s) - 4Y_2(s) &= -4 + e^{-s}\Big[\frac{64}{s^2} + \frac{64}{s}\Big] \\ Y_1(s) - (s-2)Y_2(s) &= 4 \end{aligned}\right\}$$

Solve for $Y_1(s)$ and $Y_2(s)$ and invert:-

So: multiply second equation by $-(s-2)$ and add to first:-

$$\Big[\underbrace{(s-2)^2 - 4}_{\substack{s^2 - 4s \\ \shortparallel \\ s(s-4)}}\Big]Y_2(s) = \underbrace{-4(s-2)-4}_{-4s+4} + e^{-s}\Big(\frac{64s+64}{s^2}\Big)$$

$$\Rightarrow \quad Y_2(s) = \underbrace{\frac{-4s+4}{s(s-4)}}_{\overset{\shortparallel}{\frac{A}{s} + \frac{B}{s-4}}} + e^{-s}\underbrace{\Big[\frac{64s+64}{s^3(s-4)}\Big]}_{\frac{C}{s} + \frac{D}{s^2} + \frac{E}{s^3} + \frac{F}{s-4}}$$

$$\Rightarrow \quad Y_2(s) = -\frac{1}{s} - \frac{3}{s-4} + e^{-s}\left[-\frac{5}{s} - \frac{20}{s^2} - \frac{16}{s^3} + \frac{5}{s-4}\right]$$

(check it!)

$$\Rightarrow \quad y_2(t) = -1 - 3e^{4t} + \Big[-5 - 20(t-1) - \tfrac{16}{2}(t-1)^2 + 5e^{4(t-1)}\Big]u(t-1)$$

$$= \underline{-1 - 3e^{4t} + \left[7 - 4t - 8t^2 + 5e^{4(t-1)}\right]u(t-1)}$$

Similarly (check!):—

$$Y_1(s) = \frac{2}{s} - \frac{6}{s-4} + e^{-s}\left[-\frac{10}{s} + \frac{24}{s^2} + \frac{32}{s^3} + \frac{10}{s-4}\right]$$

$$\Rightarrow \quad y_1(t) = 2 - 6e^{4t} + \Big[-10 + 24(t-1) + 16(t-1)^2 + 10e^{4(t-1)}\Big]u(t-1)$$

$$= \underline{2 - 6e^{4t} + \left[-18 - 8t + 16t^2 + 10e^{4(t-1)}\right]u(t-1)}$$

(16.12)

Summary:

1) Understand use of Laplace Transform to solve I.V.P. for 2nd order linear ODE, constant coeffs.

2) Idea of transfer function

3) Use of Laplace transform to solve (inhomogeneous) systems of coupled linear ODEs, with constant matrix A: know how to handle 2×2 cases.

4) Notice how forcing terms involving $u(t-a)$ are handled, and generally how to use Second Shifting Theorem

K § ~~5.6, 5.7, 5.3~~
6.3, 6.4, 6.7