The Convolution Theorem

In general

$$\mathcal{L}(f(t)\,g(t)) \neq \mathcal{L}(f(t))\,\mathcal{L}(g(t))$$

and

$$\mathcal{L}^{-1}(F(s)\,G(s)) \neq \mathcal{L}^{-1}(F(s))\,\mathcal{L}^{-1}(G(s))$$

However, there is an important relation between f, g, F & G, called the Convolution Theorem:

$$\boxed{\mathcal{L}^{-1}(F(s)G(s)) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau} \qquad (17.1)$$

Here τ is a dummy variable of integration. If you prefer, call it x or θ or ϕ or... —

The integral on the RHS of (17.1) is called the $\left\{\begin{array}{l}\underline{\text{convolution}}\\ \underline{\text{folding together}}\end{array}\right.$ of f and g.

Sometimes we write

$$\int_0^t f(\tau)g(t-\tau)\,d\tau = (f*g)(t) \qquad (17.2)$$

$$(\text{OR} = f(t)*g(t))$$

$*$ denotes "convolution product"

It is important to note that

$$(f*g)(t) = (g*f)(t) \qquad (17.3)$$

Pf: $(g * f)(t) = \int_0^t g(\tau) f(t-\tau)\, d\tau$

Change variable of integration: Put

$u = t - \tau \iff \tau = t - u$

$d\tau = -du$

$\tau = 0 \iff u = t \qquad \tau = t \iff u = 0$

So get:

$$(g * f)(t) = \int_t^0 g(t-u) f(u) (-du)$$

$$= \int_0^t f(u) g(t-u)\, du$$

$$= \underline{(f * g)(t)}$$

The Convolution Theorem (17.1) says:

$$\boxed{\begin{aligned} \mathcal{L}((f*g)(t)) &= \mathcal{L}((g*f)(t)) = F(s)G(s) \\ \mathcal{L}^{-1}(F(s)G(s)) &= (f*g)(t) = (g*f)(t) \end{aligned}} \quad (17.4)$$

EX 1: $H(s) = \frac{1}{s^3}$

$$= \underbrace{\frac{1}{s}}_{F(s)} \underbrace{\frac{1}{s^2}}_{G(s)}$$

$$\mathcal{L}^{-1}(F(s)) = 1 \qquad = f(t)$$

$$\mathcal{L}^{-1}(G(s)) = t \qquad = g(t)$$

$$\mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}(F(s)G(s)) = (f * g)(t)$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t 1.(t-\tau)\, d\tau$$

$$= t\int_0^t 1\, d\tau - \int_0^t \tau\, d\tau$$

$$= t\, [\tau]_{\tau=0}^{\tau=t} - \left[\tfrac{1}{2}\tau^2\right]_{\tau=0}^{\tau=t}$$

$$= t^2 - \tfrac{1}{2}t^2$$

$$= \underline{\tfrac{1}{2}t^2}$$ — as we already knew.

EX2: $H(s) = \dfrac{2s}{(s^2+1)^2}$

$$= \underbrace{\frac{2s}{s^2+1}}_{F(s)} \cdot \underbrace{\frac{1}{(s^2+1)}}_{G(s)}$$

$$\mathcal{L}^{-1}(F(s)) = 2\cos t = f(t)$$
$$\mathcal{L}^{-1}(G(s)) = \sin t = g(t)$$

$$\mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}(F(s)\,G(s)) = (f*g)(t).$$

$$= \int_0^t 2\cos(\tau)\sin(t-\tau)\,d\tau$$

$$\left[\text{Use } \; 2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)\right]$$

$$= \int_0^t [\sin(t) + \sin(t-2\tau)]\,d\tau$$

$$= \sin(t)\left[\tau\right]_{\tau=0}^{\tau=t} + \tfrac{1}{2}\left[\cos(t-2\tau)\right]_{\tau=0}^{\tau=t}$$

$$= t\sin(t) + \tfrac{1}{2}\left[\cos(-t) - \cos(t)\right]$$

$$= t\sin(t)$$

Usefulness in ODEs:

Consider again

$$y''(t) + a\,y'(t) + b\,y(t) = r(t)$$
$$y(0) = y'(0) = 0$$

$$\Rightarrow \quad Y(s) = Q(s)\,R(s)$$ (p. 16.4)

where

$$Y(s) = \mathcal{L}(y(t)), \quad R(s) = \mathcal{L}(r(t))$$
$$Q(s) = \frac{1}{s^2 + as + b}$$ transfer function

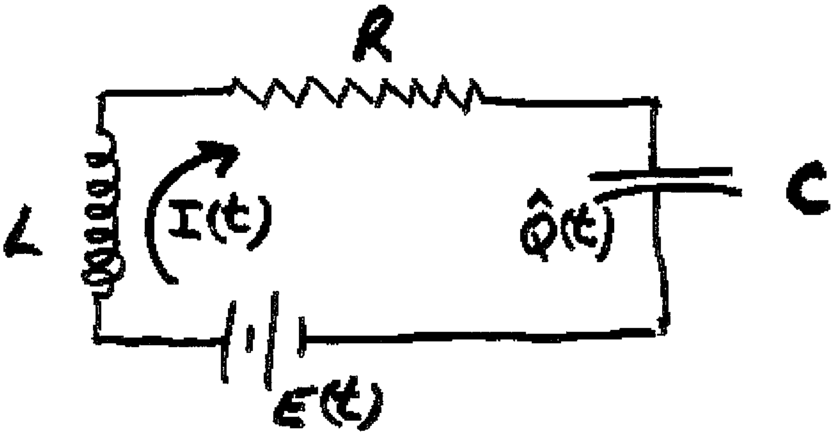
See now, by Convolution Theorem, that

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}(Q(s)\,R(s)) = (q * r)(t)$$
$$= \int_0^t q(\tau)\,r(t-\tau)\,d\tau \qquad (17.5)$$

where

$$q(t) = \mathcal{L}^{-1}(Q(s))$$

EX 3:



[Here $\hat{Q}(t)$ charge on capacitor — nothing to do with $Q(s)$!!]

As before (p. 1.8, 1.9, 12.1), we have

$$\left.\begin{aligned} L I'(t) + R I(t) + \tfrac{1}{C}\hat{Q}(t) &= E(t) \\ \hat{Q}'(t) &= I(t) \end{aligned}\right\}$$

$$\Rightarrow \quad L\hat{Q}''(t) + R\hat{Q}'(t) + \tfrac{1}{C}\hat{Q}(t) = E(t)$$

OR

$$\hat{Q}''(t) + \underbrace{\frac{R}{L}}_{a}\hat{Q}'(t) + \underbrace{\frac{1}{LC}}_{b}\hat{Q}(t) = \underbrace{\frac{1}{L}E(t)}_{r(t)}$$

Suppose also:

$$\hat{Q}(0) = 0 \quad , \quad \hat{Q}'(0) = I(0) = 0$$

According to (17.5), the solution is:–

$$\hat{Q}(t) = \int_0^t q(\tau)\,\frac{E(t-\tau)}{L}\,d\tau$$

where

$$q(t) = \mathcal{L}^{-1}(Q(s))$$
$$= \mathcal{L}^{-1}\left[\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right]$$

Form of $q(t)$ will depend on values of R, L, C.

e.g. If $R = 0$,

$$Q(s) = \frac{1}{s^2 + \omega^2}, \qquad \omega = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \quad q(t) = \frac{1}{\omega}\sin(\omega t)$$

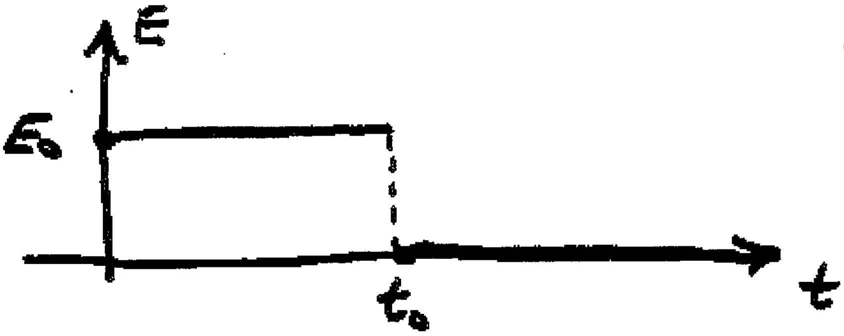
and

$$\hat{Q}(t) = \frac{1}{\omega L}\int_0^t \sin(\omega\tau)\,E(t-\tau)\,d\tau$$

$$\left(\frac{1}{\omega L} = \sqrt{\frac{C}{L}}\right)$$

(17.9)

So if



$$E(t) = \begin{cases} E_0, & 0 \le t < t_0 \\ 0, & t_0 < t \end{cases}$$

$$\Rightarrow \quad E(t-\tau) = \begin{cases} E_0, & 0 \le t-\tau < t_0 \Rightarrow t-t_0 < \tau \le t \\ 0 & t_0 < t-\tau \Rightarrow \quad \tau < t-t_0 \end{cases}$$

we have

$$\hat{Q}(t) = \frac{1}{\omega L} E_0 \int_{t-t_0}^{t} \sin(\omega\tau)\, d\tau$$

$$= -\frac{1}{\omega^2 L} E_0 \left[\cos(\omega\tau)\right]_{\tau = t-t_0}^{\tau = t}$$

$$= -E_0 C \left[\cos(\omega t) - \cos[\omega(t-t_0)]\right]$$

$$= 2E_0 C \sin\left(\frac{\omega t_0}{2}\right) . \sin\left[\omega\left(t - \frac{t_0}{2}\right)\right]$$

[using $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$]

Laplace Transform of an integral:

Since $\mathcal{L}(1) = \frac{1}{s}$, $\mathcal{L}(f(t)) = F(s)$

and

$$(f * 1)(t) = \int_0^t f(\tau) . 1 . d\tau$$

it follows from the Convolution Theorem that

$$\boxed{\begin{aligned} \mathcal{L}\left(\int_0^t f(\tau) d\tau \right) &= \frac{1}{s} F(s) \\ \mathcal{L}^{-1}\left(\frac{1}{s} F(s) \right) &= \int_0^t f(\tau) d\tau \end{aligned}} \quad (17.6)$$

EX:

$$G(s) = \frac{1}{s(s^2 - a^2)} , \quad g(t) = ?$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2 - a^2} \right) = \frac{1}{a} \sinh(at)$$

So

$$\mathcal{L}^{-1}\left(\frac{1}{s} \frac{1}{s^2 - a^2} \right) = \frac{1}{a} \int_0^t \sinh(a\tau) d\tau$$

$$= \frac{1}{a^2} \left[\cosh(a\tau) \right]_{\tau=0}^{\tau=t}$$

$$= \frac{1}{a^2}\left[\cosh(at) - 1\right]$$

EX: $G(s) = \frac{1}{s^2}\frac{1}{s^2-a^2}$, $g(t) = ?$

Integrating previous answer:-

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{1}{s^2-a^2}\right) = \int_0^t \frac{1}{a^2}\left[\cosh(a\tau)-1\right]d\tau$$

$$= \frac{1}{a^2}\left[\frac{\sinh(a\tau)}{a} - \tau\right]_{\tau=0}^{\tau=t}$$

$$= \frac{1}{a^3}\sinh(at) - \frac{t}{a^2}$$

Differentiation of Transforms:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow F'(s) = \int_0^\infty \frac{d}{ds}\left(e^{-st}\right) f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\, t\, f(t)\,dt$$

$$\boxed{\mathcal{L}(t f(t)) = -F'(s)\ ,\quad \mathcal{L}^{-1}(F'(s)) = -t f(t) \quad (17.7)}$$

<u>Ex:</u> $\mathcal{L}(\sin(\alpha t)) = \frac{\alpha}{s^2+\alpha^2}$

$\Rightarrow \mathcal{L}(t\sin(\alpha t)) = -\frac{d}{ds}\left(\frac{\alpha}{s^2+\alpha^2}\right)$

$$= \frac{2\alpha s}{(s^2+\alpha^2)^2}$$

Similarly:

$$\mathcal{L}(t\cos(\alpha t)) = -\frac{d}{ds}\left(\frac{s}{s^2+\alpha^2}\right)$$

$$= \frac{s^2-\alpha^2}{(s^2+\alpha^2)^2}$$

<u>Summary:</u>

1) Understand <u>Convolution Theorem</u> (<u>no</u> <u>proof</u>) and how to use it to evaluate transforms, and in ODEs

2) Laplace Transform of $\int_0^t f(\tau)d\tau$ equals $\frac{1}{s}F(s)$

3) Inverse of the derivative of a Laplace Transform.

KS ~~5.2, 5.4, 5.5~~ 6.2, 6.5, 6.6