Lec. 18 MATH2100/2010

Integration of Transforms

Suppose $\mathcal{L}(g(t)) = G(s)$

Let $F(s) = \int_s^\infty G(\sigma)\,d\sigma \quad \longleftrightarrow \quad f(t) = \,?$

See $F'(s) = -G(s)$

$\Rightarrow \quad \mathcal{L}^{-1}(F'(s)) = -\mathcal{L}^{-1}(G(s))$

$\Rightarrow \quad -t\,f(t) = -g(t)$

$\Rightarrow \quad f(t) = \dfrac{g(t)}{t}$

So

$$\boxed{\begin{aligned} &\mathcal{L}^{-1}\left(\int_s^\infty G(\sigma)\,d\sigma\right) = \frac{g(t)}{t} \qquad (18.1)\\ &\mathcal{L}\left(\frac{g(t)}{t}\right) = \int_s^\infty G(\sigma)\,d\sigma \end{aligned}}$$

[Trouble if $\frac{g(t)}{t}$ blows up as $t \to 0_+$.]

<u>EX:</u> $F(s) = \ln\left(1 - \frac{a^2}{s^2}\right) \leftrightarrow f(t) = ?$

First, find $G(s)$ s.t. $F(s) = \int_s^\infty G(\sigma)\,d\sigma$

$$\Rightarrow \quad F'(s) = -G(s)$$

$$\Rightarrow G(s) = -\frac{d}{ds}\ln\left(1 - \frac{a^2}{s^2}\right)$$

$$= -\frac{d}{ds}\left[\ln\left(\frac{s^2-a^2}{s^2}\right)\right]$$

$$= -\frac{d}{ds}\left[\ln(s^2-a^2) - \ln(s^2)\right]$$

$$= -\frac{d}{ds}\left\{\ln[(s-a)(s+a)] - 2\ln s\right\}$$

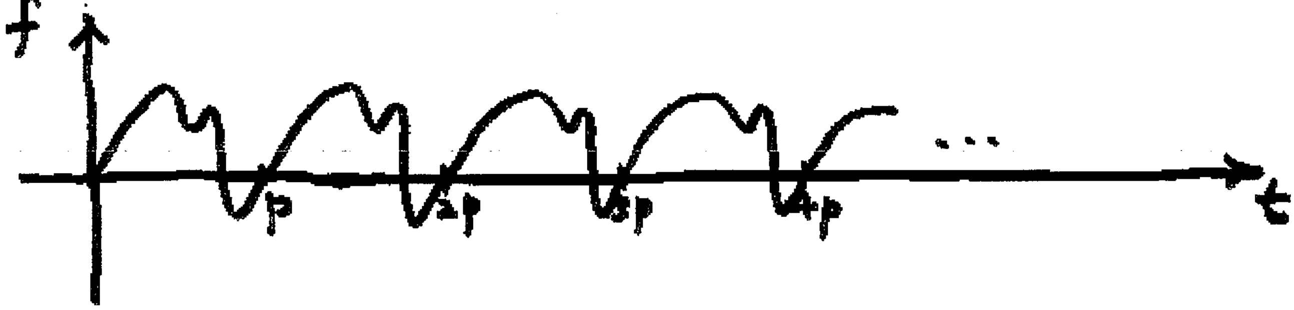
$$= -\frac{d}{ds}\left\{\ln(s-a) + \ln(s+a) - 2\ln s\right\}$$

$$= -\frac{1}{s-a} - \frac{1}{s+a} + \frac{2}{s}$$

$$\Rightarrow g(t) = -e^{at} - e^{-at} + 2 = 2 - 2\cosh(at$$

Then $f(t) = \frac{g(t)}{t} = \underline{\frac{2-2\cosh(at)}{t}}$

Transforms of Periodic Functions



We say $f(t)$ is periodic with period $p>0$:

$$f(t+p) = f(t) \qquad \forall t > 0$$

Now consider

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^p e^{-st} f(t)\,dt + \int_p^{2p} e^{-st} f(t)\,dt + \int_{2p}^{3p} e^{-st} f(t)\,dt + \cdots$$

See $\displaystyle\int_p^{2p} e^{-st} f(t)\,dt = \int_0^p e^{-s(u+p)} f(u+p)\,du$

$$= e^{-sp}\int_0^p e^{-su} f(u)\,du$$

$$= e^{-sp}\int_0^p e^{-st} f(t)\,dt$$

[Put $t = u+p \Leftrightarrow u = t-p$
$dt = du$
$t = p \Leftrightarrow u = 0$
$t = 2p \Leftrightarrow u = p$]

Similarly $$\int_{2p}^{3p} e^{-st} f(t)\,dt = e^{-2sp}\int_0^p e^{-st} f(t)\,dt$$

and so on :

$$\mathcal{L}(f(t)) = \left(\int_0^p e^{-st} f(t)\,dt\right)\left\{1 + e^{-sp} + e^{-2sp} + \dots\right\}$$

Recall $$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad |x|<1$$

$$\Rightarrow 1 + e^{-sp} + e^{-2sp} + \dots = \frac{1}{1-e^{-sp}}, \quad \begin{array}{c} s>0 \\ \Downarrow \\ 0<e^{-sp}<1 \end{array}$$

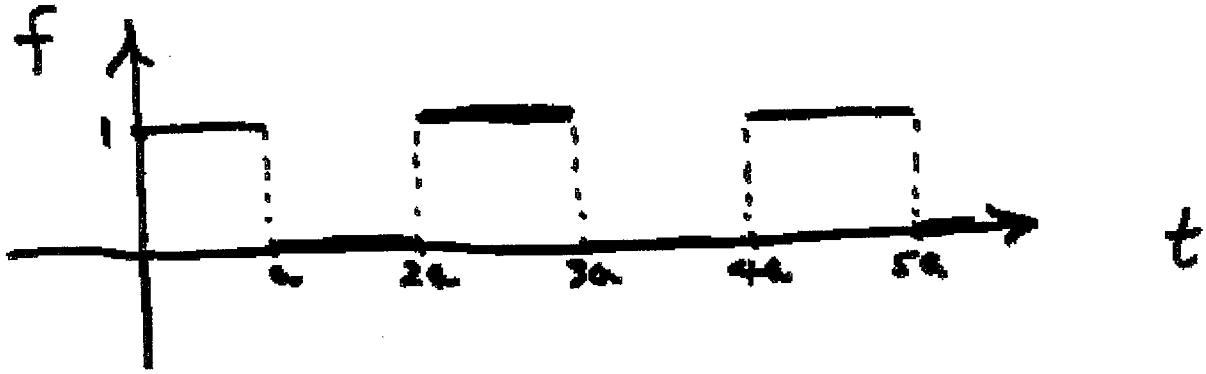
So we have

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt \quad (18.2)}$$

(18.5)

EX: $f(t) = \begin{cases} 1, & 0<t<a \\ 0, & a<t<2a \end{cases}$

and $f(t+2a) = f(t)$, $t>0$



$$\Rightarrow F(s) = \mathcal{L}(f(t)) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{a} e^{-st} .1.\, dt$$

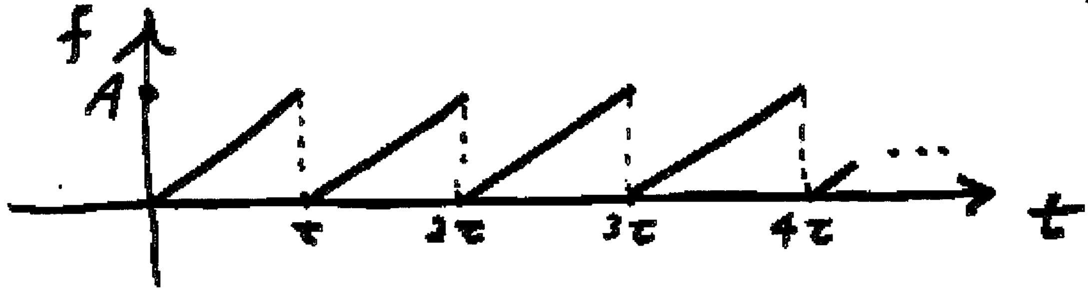
$$= \frac{1}{1-e^{-2as}} \left[\frac{e^{-st}}{-s} \right]_{t=0}^{t=a}$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{e^{-as}-1}{-s} \right]$$

$$= \frac{1}{\cancel{(1-e^{-as})}(1+e^{-as})} \frac{\cancel{(1-e^{-as})}}{s}$$

$$= \frac{1}{s} \frac{1}{(1+e^{-as})}$$

EX:



$$f(t) = \frac{At}{\tau}\ , \quad 0 \leq t < \tau$$

and

$$f(t+\tau) = f(t)\ , \quad t > 0$$

$\Rightarrow$

$$F(s) = \frac{1}{1-e^{-\tau s}} \int_0^{\tau} e^{-st}\, \frac{At}{\tau}\, dt$$

$$= \frac{A}{\tau}\, \frac{1}{1-e^{-\tau s}} \left\{ \left[\frac{t e^{-st}}{-s}\right]_{t=0}^{t=\tau} + \frac{1}{s}\int_0^{\tau} e^{-st} dt \right\}$$

$$= \frac{A}{\tau}\, \frac{1}{1-e^{-\tau s}} \left\{ -\frac{\tau}{s} e^{-s\tau} - \frac{1}{s^2}\left[e^{-st}\right]_{t=0}^{t=\tau} \right\}$$

$$= \frac{A}{\tau}\, \frac{1}{1-e^{-\tau s}} \left\{ -\frac{\tau}{s} e^{-s\tau} \underbrace{- \frac{1}{s^2} e^{-s\tau} + \frac{1}{s^2}}_{\frac{1}{s^2}(1-e^{-\tau s})} \right\}$$

$$= A \left\{ \frac{1}{\tau s^2} - \frac{e^{-\tau s}}{s(1-e^{-\tau s})} \right\}$$

Relations at $t=0$, $s=\infty$ and $s=0$, $t=\infty$.

Recall

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

As $s \to \infty$, the integral can be expected to go to zero.

So

$$\boxed{\begin{array}{c} \lim_{s\to\infty} sF(s) = f(0) \\ \text{OR} \\ \lim_{s\to\infty} sF(s) = \lim_{t\to 0_+} f(t) \end{array}} \quad (18.3)$$

EX: $f(t) = e^{at}$ $\qquad F(s) = \frac{1}{s-a}$

$$\lim_{t\to 0_+} f(t) = e^0 = 1 \qquad \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s}{s-a}$$

$$= 1$$

EX: $f(t) = \sin(\alpha t)$ $\qquad F(s) = \dfrac{\alpha}{s^2+\alpha^2}$

$\lim_{t \to 0_+} f(t) = 0 \qquad \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \dfrac{\alpha s}{s^2+\alpha^2}$

$$= 0$$

Now consider again

$$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

As $s \to 0_+$, the integral goes to ...

$$\int_0^\infty f'(t)\,dt = \left[f(t)\right]_{t=0}^{t=\infty} = f(\infty) - f(0)$$

provided $f(\infty)$ is finite.

So we have

$$f(\infty) - f(0) = \lim_{s\to 0_+} sF(s) - f(0)$$

$$\boxed{\begin{aligned} &\lim_{s\to 0_+} sF(s) = f(\infty) \\ &\underline{\text{OR}} \\ &\lim_{s\to 0_+} sF(s) = \lim_{t\to\infty} f(t) \end{aligned}} \qquad (18.9)$$

EX: $f(t) = 3 - e^{-at}$ $(a>0)$

$$F(s) = \frac{3}{s} - \frac{1}{s+a}$$

$$sF(s) = 3 - \frac{s}{s+a}$$

$$\lim_{t\to\infty} f(t) = 3 \ , \quad \lim_{s\to 0_+} sF(s) = 3$$

EX: $f(t) = \cos(\alpha t)$ $\quad F(s) = \dfrac{s}{s^2+\alpha^2}$

$$sF(s) = \frac{s^2}{s^2+\alpha^2}$$

$$\lim_{t\to\infty} \cos(\alpha t) = \ ? \qquad \lim_{s\to 0_+} sF(s) = 0$$

No result as $\lim_{t\to\infty} f(t)$ undefined.

Behaviour of $f(t)$ near $t=0$ is related to behaviour of $sF(s)$ near $s=\infty$, and vice versa.

One more result:

Recall

$$\mathcal{L}(t\,f(t)) = -F'(s)$$

$$\Rightarrow \quad \mathcal{L}(t^2 f(t)) = (-1)^2 F''(s)$$

$$\Rightarrow \quad \mathcal{L}(t^3 f(t)) = (-1)^3 F'''(s)$$

$$\vdots$$

$$\boxed{\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)} \qquad (18.5)$$

i.e.

$$\int_0^\infty e^{-st} t^n f(t)\,dt = (-1)^n F^{(n)}(s)$$

Let $s \to 0$: <u>If</u> integral stays finite, we have

$$\boxed{\underbrace{\int_0^\infty t^n f(t)\,dt}_{\substack{n\text{-th} \\ \text{moment} \\ \text{of } f}} = (-1)^n \lim_{s\to 0} F^{(n)}(s) = (-1)^n F^{(n)}(0)} \qquad (18.6)$$

Can be very useful if can find $F(s)$ but not $f(t)$.

Summary:

1) Integration of Laplace Transforms
- can use to get new results.

2) Laplace Transform of Periodic Functions
- know how to derive (18.2) and be able to calculate $F(s)$ given periodic $f(t)$.

3) Know that when all limits exist, we have

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0_+} f(t)$$

$$\lim_{s \to 0_+} sF(s) = \lim_{t \to \infty} f(t)$$

4) Be aware of (18.6)

K pp ~~290, 301~~. 248, 269

Review Ch. #6