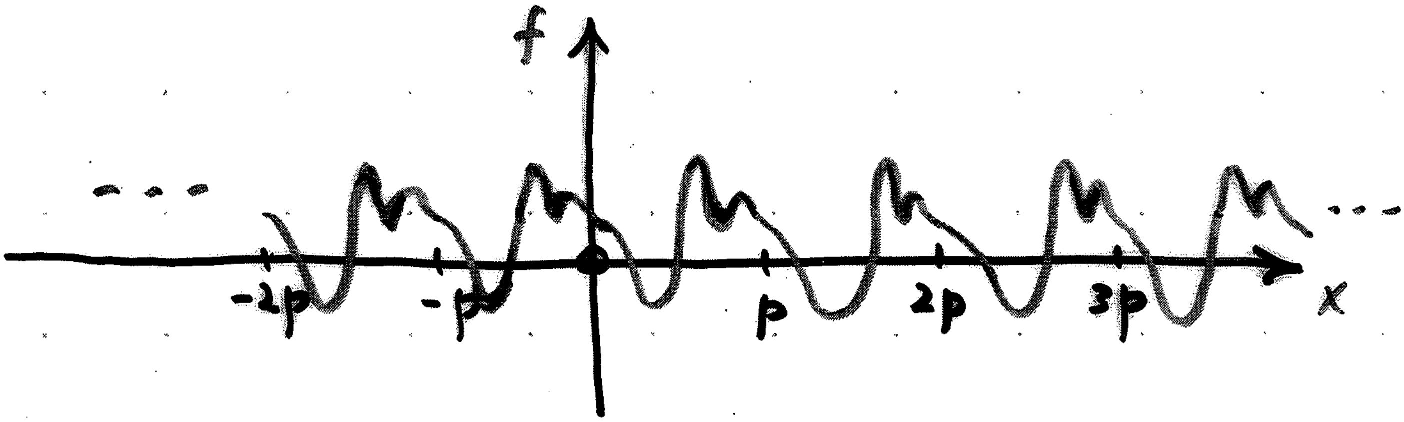
Lec. 19 MATH2100 (= Lec. 1 MATH2011)

Consider a periodic function $f(x)$ defined for all x, with period $p > 0$.

$$f(x+p) = f(x), \qquad -\infty < x < \infty$$



Fourier's Idea: Analyse in terms of the elementary functions with period p:

$$1, \cos\left(\frac{2\pi x}{p}\right), \cos\left(\frac{4\pi x}{p}\right), \ldots$$

$$\sin\left(\frac{2\pi x}{p}\right), \sin\left(\frac{4\pi x}{p}\right), \ldots$$

The birth of Harmonic Analysis!
Contrast with idea of synthesis.

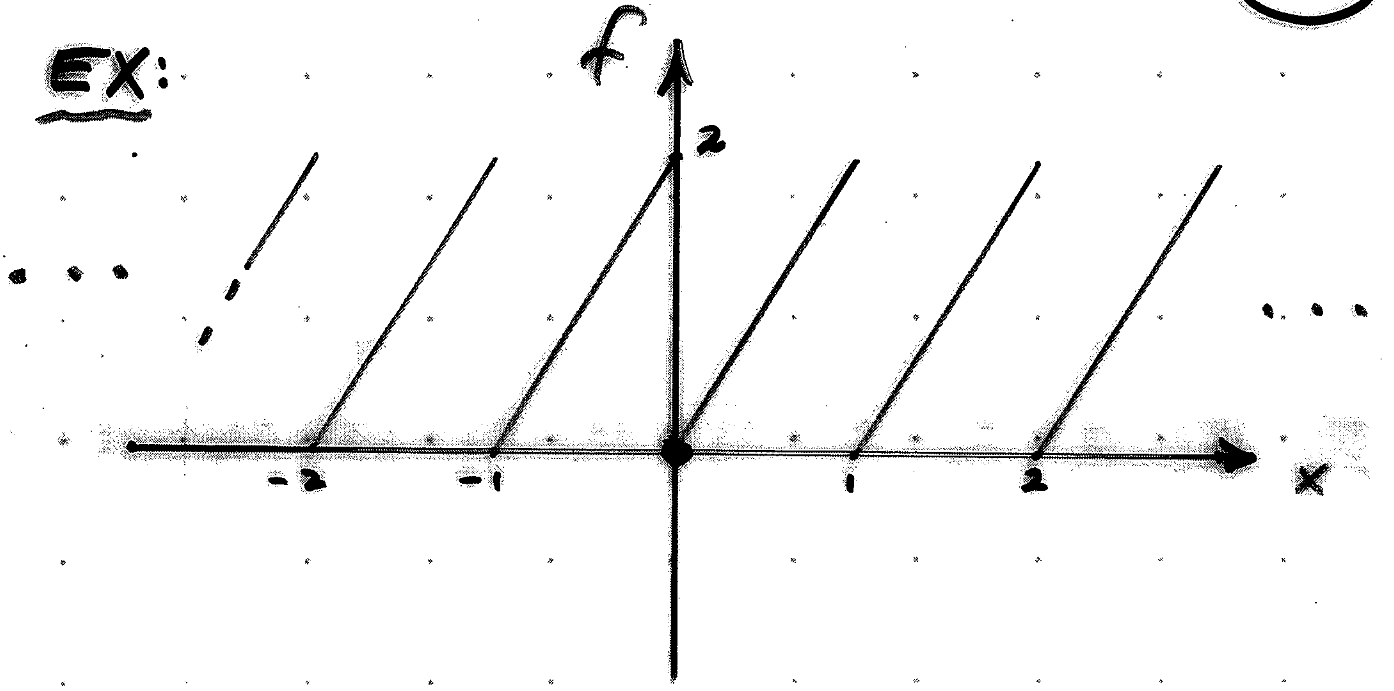
So Fourier wrote:

$$\left.\begin{aligned} f(x) = {} & a_0 + a_1 \cos\left(\frac{2\pi x}{p}\right) + b_1 \sin\left(\frac{2\pi x}{p}\right) \\ & + a_2 \cos\left(\frac{4\pi x}{p}\right) + b_2 \sin\left(\frac{4\pi x}{p}\right) \\ & + \dots \qquad \dots \end{aligned}\right\} \quad (19.1)$$

and found how to calculate the coefficients $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ to make it work.

The RHS of (19.1) is called the <u>Fourier Series</u> (expansion) of $f(x)$, and $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are called the corresponding <u>Fourier coefficients</u>.

EX:



$$f(x) = 2x \,, \quad 0 \leq x < 1$$

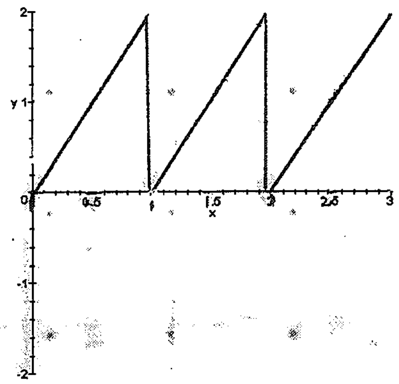
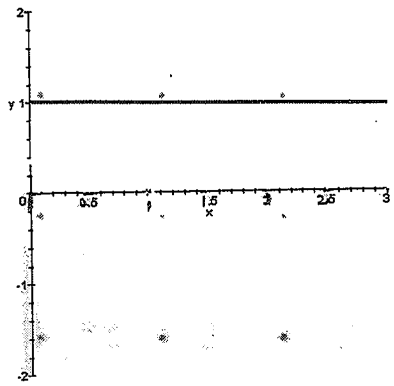
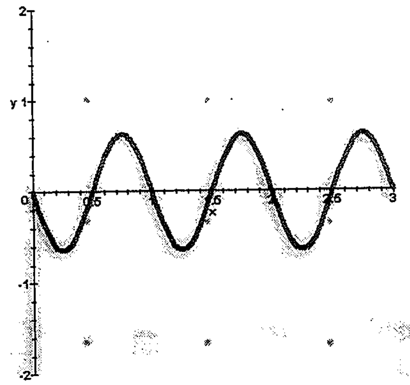
$$f(x+1) = f(x) \,, \quad -\infty < x < \infty$$

This function is defined for <u>all</u> $-\infty < x < \infty$, and is periodic with period 1. It is <u>piecewise</u> <u>continuous</u>.

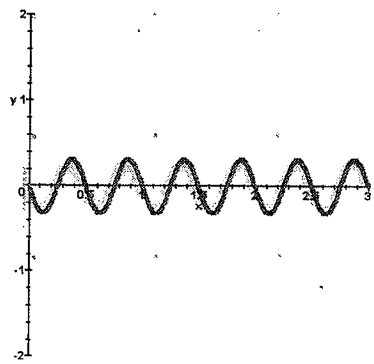
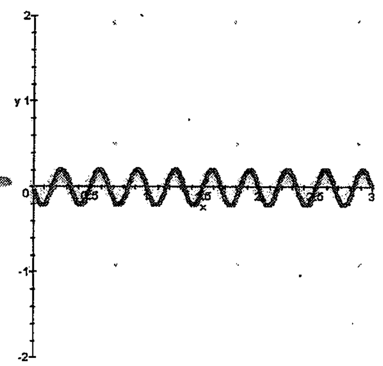
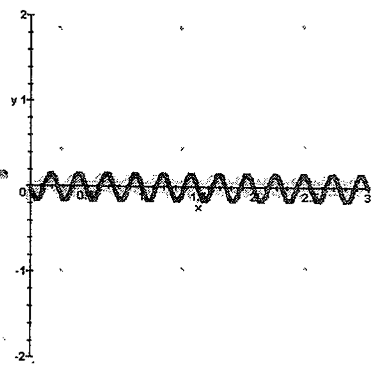
In this case, the relevant elementary functions have period 1 : They are

$1, \cos(2\pi x), \cos(4\pi x), \ldots$ and $\sin(2\pi x), \sin(4\pi x), \ldots$

The Fourier analysis of $f(x)$ is:

 =  + 

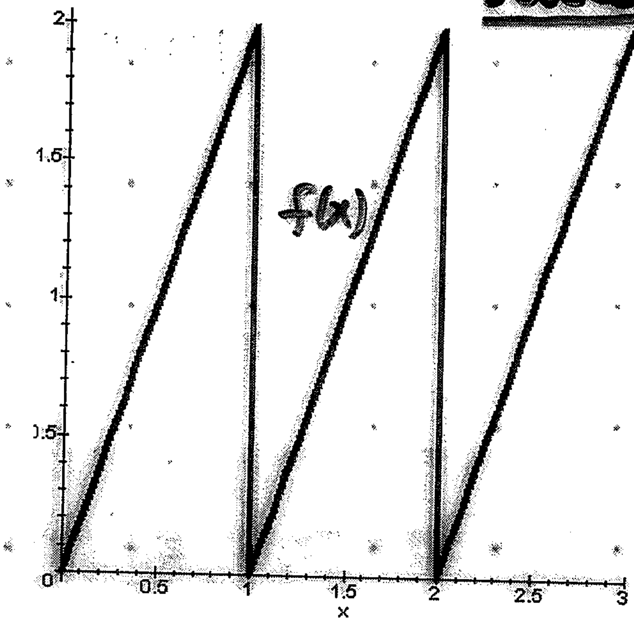
$f(x)$ $\qquad$ 1 $\qquad$ $-\frac{2}{\pi}\sin(2\pi x)$

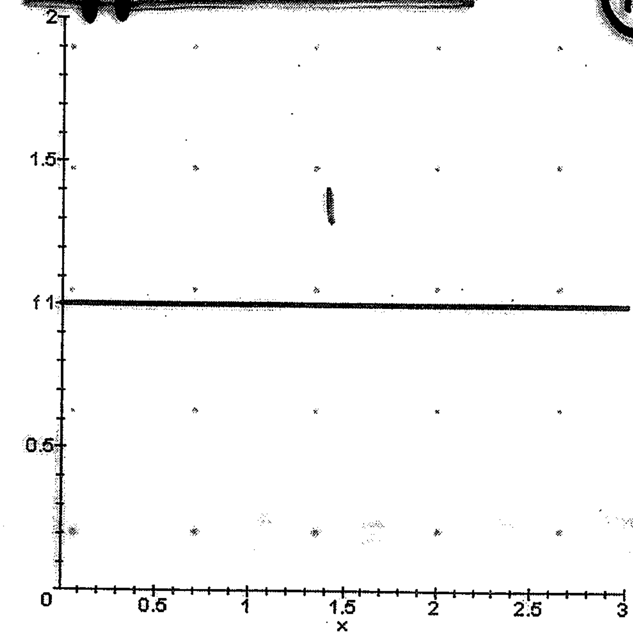
+  +  + 

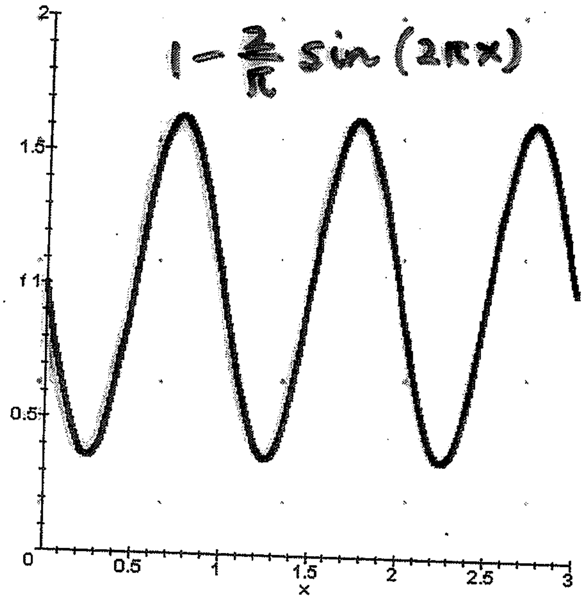
$-\frac{1}{\pi}\sin(4\pi x)$ $\qquad$ $-\frac{2}{3\pi}\sin(6\pi x)$ $\qquad$ $-\frac{1}{2\pi}\sin(8\pi x)$

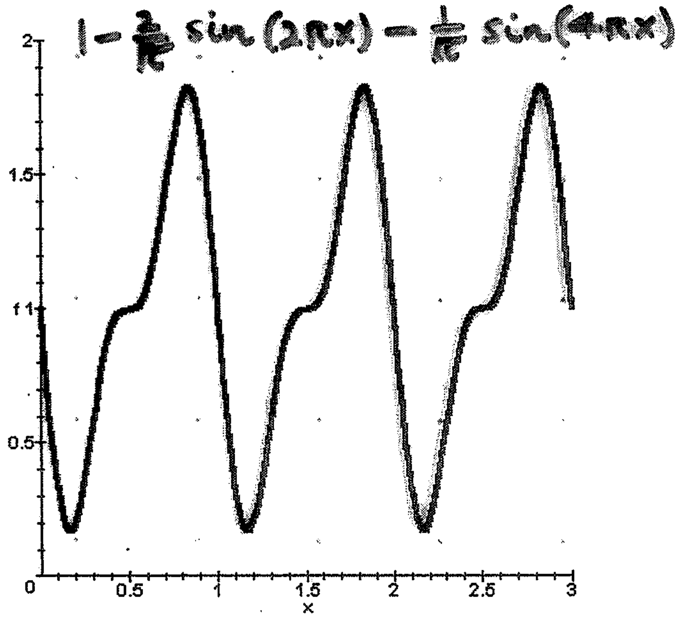
$+ \ldots$

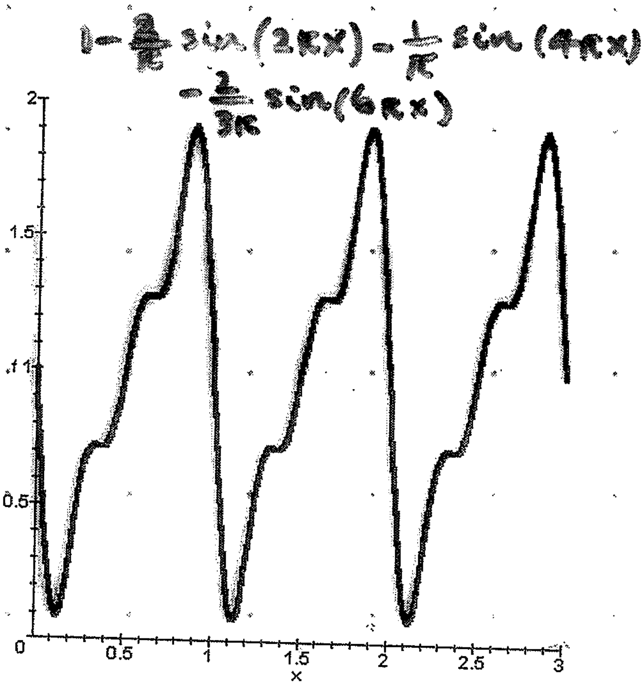
Successive approximations:

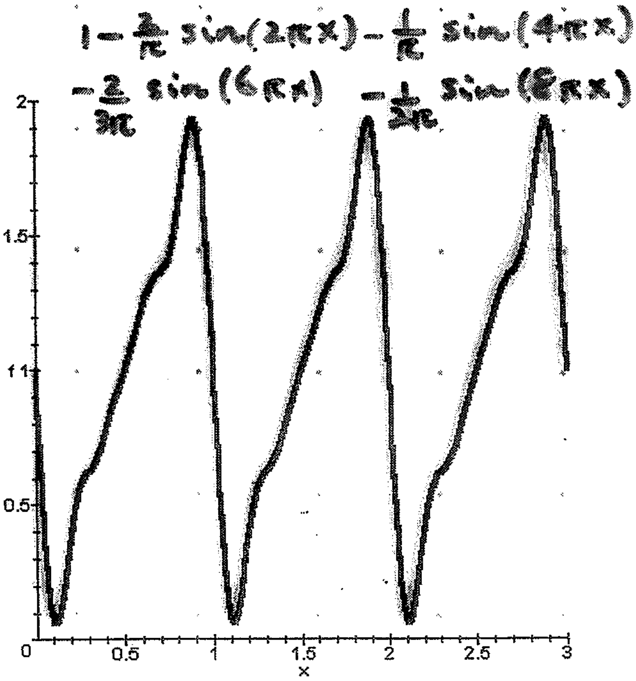












In general, given $f(x)$, $-\infty < x < \infty$,
with period p

(i.e. $f(x+p) = f(x)$, $-\infty < x < \infty$),

we try to expand as

$$f(x) = a_0 + a_1 \cos\left(\frac{2\pi x}{p}\right) + a_2 \cos\left(\frac{4\pi x}{p}\right) + \cdots$$

$$+ b_1 \sin\left(\frac{2\pi x}{p}\right) + b_2 \sin\left(\frac{4\pi x}{p}\right) + \cdots$$

OR

Setting $p = 2L$ for convenience,

$$f(x) = a_0 + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$

$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

OR

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \quad (19.2)}$$

The Fourier analysis of $f(x)$
The Fourier (series) expansion of $f(x)$

The first and most important question facing us is:

How do we find the Fourier coefficients $a_0, a_1, a_2 \dots$ $b_1, b_2 \dots$ **when we are given $f(x)$?**

[The second and more difficult question is: When does it all work? That is, when can we find $a_0, a_1, a_2 \dots$ $b_1, b_2 \dots$ such that the RHS of (19.2) equals (converges to) the LHS at each value of x? We postpone discussion of this second question until later.]

To answer the first question, we make use of orthogonality relations amongst the elementary periodic functions:—

$$\boxed{1, \quad \cos\left(\frac{n\pi x}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right) \quad (n = 1, 2, 3 \dots)} \quad (19.3)$$

Consider $\displaystyle\int_{-L}^{L} 1.\cos\left(\frac{n\pi x}{L}\right) dx$

Integrate over any one period.

$$= \frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_{x=-L}^{x=L}$$

$$= \frac{L}{n\pi}\left[\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(-n\pi)}_{0}\right]$$

$$= 0$$

Consider $\displaystyle\int_{-L}^{L} 1.\sin\left(\frac{n\pi x}{L}\right) dx = -\frac{L}{n\pi}\left[\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}$

$$= -\frac{L}{n\pi}\left[\cancel{\cos(n\pi)} - \cancel{\cos(-n\pi)}\right]$$

Cancel because $\cos\theta = \cos(-\theta)$

$$= 0$$

Consider $\displaystyle\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx \qquad (m \neq n)$

Use $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

So we have $\frac{1}{2}\int_{-L}^{L}\left\{\cos\left[\frac{(m+n)\pi x}{L}\right]+\cos\left[\frac{(m-n)\pi x}{L}\right]\right\}dx$

$$= \frac{1}{2}\left\{ \frac{L}{(m+n)\pi}\left[\cancel{\sin\left[\frac{(m+n)\pi x}{L}\right]}\right]_{x=-L}^{x=L}\right.$$

Note $m \neq n$ here

$$\left. + \frac{L}{(m-n)\pi}\left[\cancel{\sin\left[\frac{(m-n)\pi x}{L}\right]}\right]_{x=-L}^{x=L}\right\}$$

$= 0$

Consider $\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (m \neq n)$

Use $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

to get

$$\frac{1}{2}\int_{-L}^{L}\left\{\cos\left[\frac{(m-n)\pi x}{L}\right]-\cos\left[\frac{(m+n)\pi x}{L}\right]\right\}dx$$

$$= \frac{1}{2}\left\{ \frac{L}{(m-n)\pi}\left[\sin\left[\frac{(m-n)\pi x}{L}\right]\right]_{x=-L}^{x=L} - \frac{L}{(m+n)\pi}\left[\sin\left[\frac{(m+n)\pi x}{L}\right]\right]_{x=-L}^{x=L} \right\}$$

$$= 0$$

Consider $\displaystyle\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx \quad (m \neq n)$

Use $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

to get

$$\frac{1}{2}\int_{-L}^{L}\left\{\sin\left[\frac{(m+n)\pi x}{L}\right] + \sin\left[\frac{(m-n)\pi x}{L}\right]\right\} dx$$

$$= -\frac{1}{2}\left\{\frac{L}{(m+n)\pi}\left[\cos\left[\frac{(m+n)\pi x}{L}\right]\right]_{x=-L}^{x=L} + \frac{L}{(m-n)\pi}\left[\cos\left[\frac{(m-n)\pi x}{L}\right]\right]_{x=-L}^{x=L}\right\}$$

$$= 0 + 0 \quad (\text{as } \cos\theta = \cos(-\theta))$$

$$= 0$$

Finally, consider

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_{-L}^{L} \sin\left(\frac{2m\pi x}{L}\right) dx$$

$$= -\frac{1}{2}\left(\frac{L}{2m\pi}\right)\underbrace{\cancel{\left[\cos\left(\frac{2m\pi x}{L}\right)\right]_{x=-L}^{x=L}}}_{0}$$

$$= 0$$

We say that each one of the functions

$1,\ \cos\left(\frac{\pi x}{L}\right),\ \cos\left(\frac{2\pi x}{L}\right) \ \ldots \ \ \sin\left(\frac{\pi x}{L}\right),\ \sin\left(\frac{2\pi x}{L}\right) \ \ldots$

is orthogonal to every other one.
(OR they are mutually orthogonal)

$$\int_{-L}^{L} f_1(x) f_2(x)\, dx = 0$$

[Compare with orthogonal vectors:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \ldots$$

$$\mathbf{i} \cdot \mathbf{j} = 0, \quad \mathbf{i} \cdot \mathbf{k} = 0, \quad \mathbf{j} \cdot \mathbf{k} = 0 \quad \text{etc.}$$

Now compare (19.2) with writing an arbitrary vector $\mathbf{u}$ as

$$\mathbf{u} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} + \cdots \quad .$$]

Summary:

1) Idea of analysing a function with period p into a series of elementary functions with period p.

2) Orthogonality of elementary periodic functions with a given period $p = 2L$.

K § 11.1, 11.2 pp. 477 – 483, 487 – 489

K §10.1 ; pp 530, 531, 534