Qu: Why more on differential equations (DEs)?

Ans: The main tool used in mathematical modelling of real-world phenomena.

Used in physics, chemistry, all types of engineering, economics, human-movement studies, biology, ...

DEs studied and applied for more than 300 years.
The theory of DEs is still an active research area for pure mathematicians.

DEs come in two basic types:–

(1) Ordinary DEs (ODEs)

(MT152 or MATH1052; MT250 or MATH2000)

→ first-half of MATH2100 (= MATH2010)

(2) Partial DEs (PDEs)

second-half of MATH2100 (= MATH2011)

Solving DEs on computer:

MATH 2200 → MATH 3201 → ...

Using DEs in modelling:

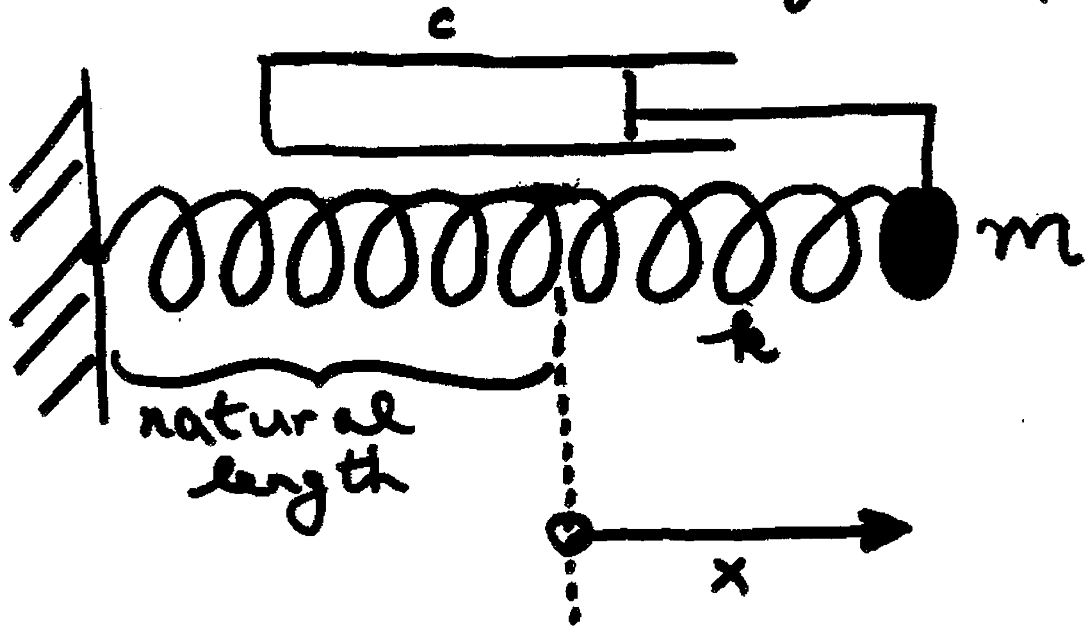
PHYS 2100, MATH3101, MATH3102, MATH3104 ....

Theory of DEs:

MATH3101, MATH3403, ...

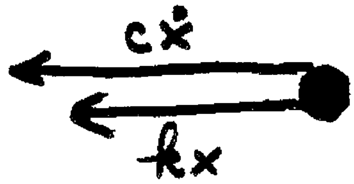
TOPIC 1: SYSTEMS of ODEs (K ch.2)

Consider mass-spring-damper:



c: damping constant   k: spring constant

Free-body diagram



$$\left(\dot{x} \equiv \frac{dx}{dt} \right)$$

$(\Leftrightarrow \underset{\sim}{F} = -kx\underset{\sim}{i} - c\dot{x}\underset{\sim}{i})$

(Linear) momentum $\quad p = m\dot{x} \quad (1.1)$

$(\Leftrightarrow \underline{p} = m\dot{x}\underline{i})$

<u>Newton's 2nd Law:-</u>

$$\frac{d\underline{p}}{dt} = \underline{F}$$

$$\Rightarrow \quad \frac{dp}{dt} = -kx - c\dot{x} \qquad (1.2)$$

Usually, we sub. (1) in (2) at once to get

$$m\ddot{x} = -kx - c\dot{x} \qquad (1.3)$$

but there are some advantages to considering (1.1) and (1.2) as a pair.

Two unknown functions: $x(t)$, $p(t)$

$$\left.\begin{aligned} m\frac{dx(t)}{dt} &= p(t) \\ \frac{dp(t)}{dt} &= -kx(t) - c\dot{x}(t) \end{aligned}\right\} \quad (1.4)$$

(1.3) is one, 2nd-order ODE for unknown $x(t)$.

(1.4) is $\left\{\begin{matrix}\text{system}\\\text{pair}\end{matrix}\right.$ of 2 coupled, 1st-order ODEs for unknowns $x(t)$, $p(t)$.

[Order = order of highest derivative
t = independent variable
$x(t)$, $p(t)$: unknown functions (dependent variables)]

In general, suppose given n-th order ODE:

$$y^{(n)}(t) = F(t, y, y', \dots, y^{(n-1)}) \qquad (1.5)$$

set $y_1 = y$, $y_2 = y'$, $\dots$ $y_n = y^{(n-1)}$

and we get the system of n 1st-order ODEs: (see K pp ~~156-7~~ 134-5)

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \dots y_n) \end{aligned}\right\} \quad (1.6)$$

(This may not be only, or most interesting, way to do it!)

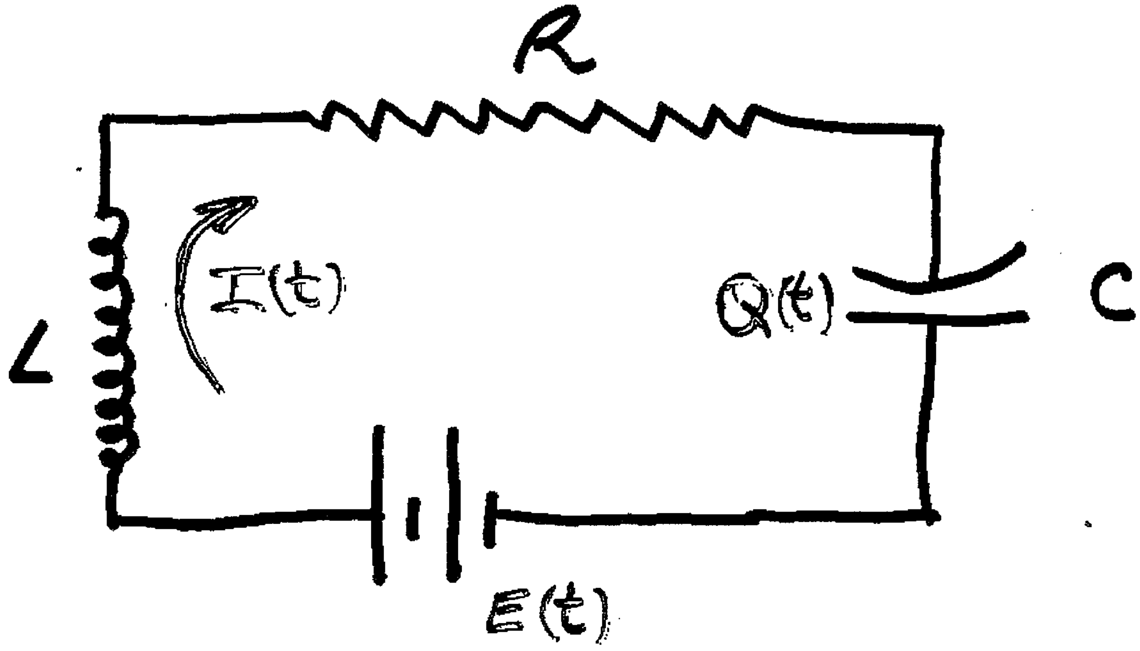
Why bother?

1) Systems arise naturally and can't always easily be written in terms of a single ODE of higher order.

2) The unknown functions in the system may have physical meanings, so the system may be giving more or other information.

Another example:

Consider LRC - circuit :-



Let $I(t)$ be current flowing at time t
$Q(t)$ be charge on capacitor at time t
$E(t)$ be applied emf at time t.

Then Kirchhoff's Law says:

$$\underbrace{L\frac{dI(t)}{dt}}_{\substack{\text{volt. drop}\\ \text{across } L}} + \underbrace{RI(t)}_{\substack{\text{volt. drop}\\ \text{across } R}} + \underbrace{\frac{Q(t)}{C}}_{\substack{\text{volt.}\\ \text{drop}\\ \text{across } C}} = \underbrace{E(t)}_{\substack{\text{applied}\\ \text{emf}}} \qquad (1.7)$$

Also we have (by charge conservation)

$$\frac{dQ(t)}{dt} = I(t) \qquad (1.8)$$

See (1.7)+(1.8) are two coupled 1st-order ODEs

Could immediately sub. (1.8) in (1.7) to get a single 2nd-order ODE, but see (1.7) and (1.8) have independent physical meaning.

Lecture Summary:

~~[Read K 1.1, 1.2, 1.4, 2.1, 2.2, 2.5, 2.8, 2.9, 2.10, 2.12, 3.0, 3.1]~~

1) DEs are principal tool for real-world modelling.

2) DEs come in 2 basic types $\left\langle \begin{array}{l} \text{ODEs} \\ \text{PDEs} \end{array}\right.$

   corresp. to ~~first and second halves of course~~ MATH 2010 and MATH 2011

3) An n-th order ODE for one unknown can be rewritten as a system of n coupled 1st order ODEs for n unknowns (Know how to do it!)

4) Systems of ODEs are interesting in their own right.

[Read K 1.1, 1.2, 1.3, 1.5, 2.1, 2.4, 2.2, 2.9]