Lec. 20 MATH2100 (= Lec. 2 MATH2011)

We saw that the functions

$$\boxed{1,\ \cos\left(\frac{\pi x}{L}\right),\ \cos\left(\frac{2\pi x}{L}\right) \dots,\ \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right)\dots} \qquad (20.1)$$

are mutually orthogonal.

i.e. if $f_1(x)$, $f_2(x)$ are any two different functions from this set, then

$$\int_{-L}^{L} f_1(x) f_2(x)\,dx = 0$$

could replace by $\int_a^{a+2L} \cdots$, any a

So now: Want for any given $f(x)$, with $f(x+2L) = f(x)$ for all x, that

$$f(x) = a_0 + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots \qquad (20.2)$$

Multiply both sides of this Eqn (20.2) by any function from the set (20.1), and integrate from $-L$ to L. See only one corresponding term on RHS survives!

So:

$$\int_{-L}^{L} 1.f(x)\,dx = a_0\int_{-L}^{L} 1.1.\,dx + 0 + 0 + \cdots$$

$$= 2L\,a_0$$

$$\Rightarrow \boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx} \quad (20.3)$$

Next:

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)f(x)\,dx = 0 + 0 + \cdots + a_n\int_{-L}^{L}\left[\cos\left(\frac{n\pi x}{L}\right)\right]^2 dx + 0 + 0 + \cdots$$

$$\Rightarrow \boxed{a_n = \frac{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)f(x)\,dx}{\int_{-L}^{L}\left[\cos\left(\frac{n\pi x}{L}\right)\right]^2 dx}} \quad (20.4)$$

<u>Next</u>:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx = 0 + 0 + 0 \dots + 0 + \dots + b_n \int_{-L}^{L} \left[\sin\left(\frac{n\pi x}{L}\right)\right]^2 dx + 0 + 0 \dots$$

$$\Rightarrow \boxed{b_n = \frac{\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx}{\int_{-L}^{L} \left[\sin\left(\frac{n\pi x}{L}\right)\right]^2 dx}} \quad (20.5)$$

In (20.4) and (20.5), n could be 1 or 2 or 3 ...

<u>Simplify</u>:

$$\int_{-L}^{L} \left[\cos\left(\frac{n\pi x}{L}\right)\right]^2 dx = \frac{1}{2} \int_{-L}^{L} \left[\cos\left(\frac{2n\pi x}{L}\right) + 1\right] dx$$

(using $\cos^2\theta = \frac{1}{2}(\cos(2\theta) + 1)$)

$$= \frac{1}{2}\left[\left(\frac{L}{2n\pi}\right) \cancel{\sin\left(\frac{2n\pi x}{L}\right)} + x\right]_{x=-L}^{x=L}$$

$$= L$$

Also: $$\int_{-L}^{L}\left[\sin\left(\frac{n\pi x}{L}\right)\right]^2 dx = \frac{1}{2}\int_{-L}^{L}\left[1-\cos\left(\frac{2n\pi x}{L}\right)\right]dx$$

(using $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$)

$$= \frac{1}{2}\left[x - \left(\frac{L}{2n\pi}\right)\cancel{\sin\left(\frac{2n\pi x}{L}\right)}^{0}_{0}\right]_{x=-L}^{x=L}$$

$$= L$$

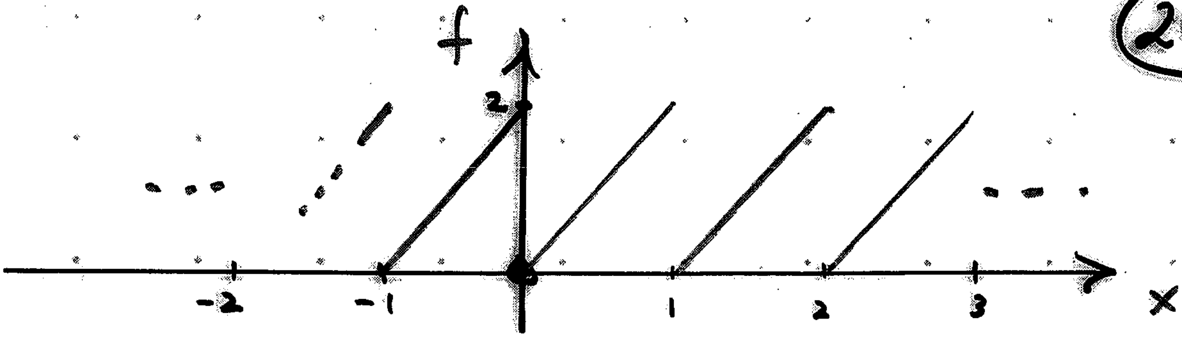
So we have:

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\,, \quad a_n = \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)f(x)\,dx \quad b_n = \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)f(x)\,dx \qquad (20.6) \quad n = 1, 2, 3, \ldots}$$

So now we know how to construct the Fourier series (i.e. do the Fourier analysis) for any given periodic function $f(x)$.

20.5

EX:



$$f(x) = 2x \quad , \quad 0 \le x < 1$$

$$f(x+1) = f(x) \quad , \quad -\infty < x < \infty$$

$\underline{p = 2L = 1}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{0}^{2L} f(x)\,dx = 1\int_0^1 2x\,dx$$

$$= \left[x^2\right]_0^1 = 1$$

$$\boxed{a_0 = 1}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx = \frac{1}{L}\int_0^{2L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx$$

$$= 2\int_0^1 \cos(2n\pi x)\, 2x\,dx$$

$$= 4\int_0^1 x\cos(2n\pi x)\,dx$$

20.6

Now $$\int \underbrace{x}_{u} \underbrace{\cos(\alpha x)}_{v'}\,dx = \underbrace{x}_{u} \underbrace{\frac{\sin(\alpha x)}{\alpha}}_{v} - \int \underbrace{1}_{u'} \underbrace{\frac{\sin(\alpha x)}{\alpha}}_{v}\,dx$$

$$= \frac{1}{\alpha} x \sin(\alpha x) + \frac{1}{\alpha^2}\cos(\alpha x) + c$$

So

$$a_n = 4\left[\frac{1}{2n\pi} x \cancel{\sin(2n\pi x)} + \frac{1}{(2n\pi)^2}\cos(2n\pi x) + c\right]_0^1$$

$$= 0 \qquad \text{as } \cos(2n\pi) = \cos(0) = 1$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx = \frac{1}{L}\int_0^{2L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx$$

$$= 2\int_0^1 \sin(2n\pi x)\, 2x\,dx$$

$$= 4\int_0^1 x \sin(2n\pi x)\,dx$$

Now $$\int x\sin(\alpha x)\,dx = -x\frac{\cos(\alpha x)}{\alpha} + \int \frac{\cos(\alpha x)}{\alpha}\,dx$$

$$= -\frac{1}{\alpha} x \cos(\alpha x) + \frac{1}{\alpha^2}\sin(\alpha x) + c$$

So

$$b_n = 4\left[-\frac{1}{2n\pi}\, x \cos(2n\pi x) + \frac{1}{(2n\pi)^2} \cancel{\sin(2n\pi x)} \right]_0^1$$

$$= 4 \cdot \left(-\frac{1}{2n\pi} \right) \cdot 1$$

$$= -\frac{2}{n\pi} \qquad \boxed{b_n = -\frac{2}{n\pi}}$$

So:

$$f(x) = a_0 + a_1 \cos(2\pi x) + a_2 \cos(4\pi x) + \dots + b_1 \sin(2\pi x) + b_2 \sin(4\pi x) + \dots$$

$$= 1 - \frac{2}{\pi} \sin(2\pi x) - \frac{2}{2\pi} \sin(4\pi x) - \frac{2}{3\pi} \sin(6\pi x) \dots$$

$$= 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\pi x)$$

as on pp. 19.4, 19.5

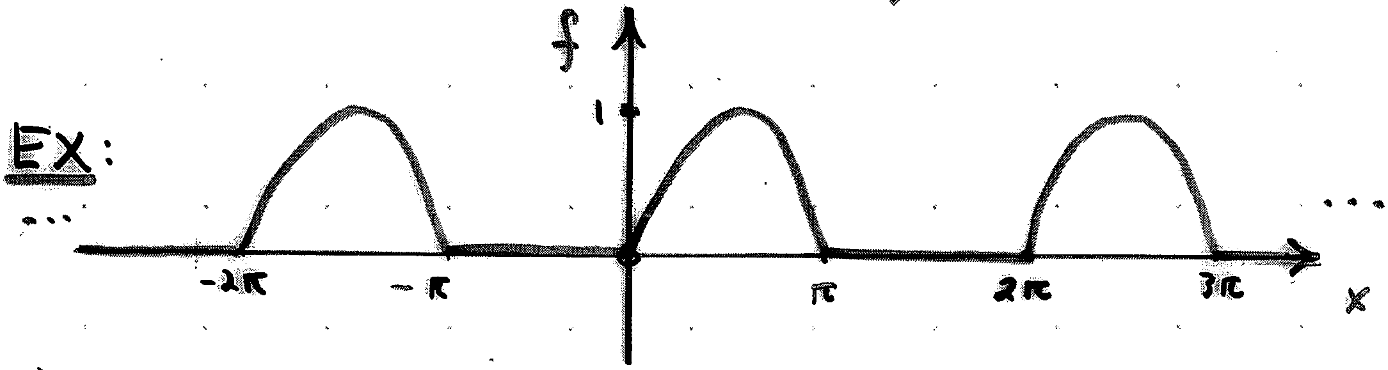
EX: $f(x) = 3\cos(17x) + 7\sin\left(\frac{15x}{2}\right)$ $(-\infty < x < \infty)$

? Fourier series: $f(x) =$ ____________

EX: $f(x) = 4\cos(5x) + 6\sin(2\pi x)$ $(-\infty < x < \infty)$

? Fourier series: ____________

EX:



"Half sine-wave rectifier"

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin(x), & 0 < x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x), \quad -\infty < x < \infty$$

$\Rightarrow$ $p = 2\pi$ so $L = \pi$

Fourier Series: 20.9

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 . f(x)\, dx = \frac{1}{2\pi} \int_0^{\pi} \sin x \, dx$$

$$= -\frac{1}{2\pi} \left[\cos x \right]_0^{\pi} = -\frac{1}{2\pi} \left[-1 - 1 \right] = \frac{1}{\pi}$$

$$\boxed{a_0 = \frac{1}{\pi}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} \cos(nx) \sin(x)\, dx$$

(Use $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B) \right]$)

$$\Rightarrow a_n = \frac{1}{2\pi} \int_0^{\pi} \left\{ \sin[(1+n)x] + \sin[(1-n)x] \right\} dx \qquad (*)$$

Now: care with case $n = 1$.

$n \neq 1$ gives $a_n = -\frac{1}{2\pi} \left[\frac{1}{(1+n)} \cos[(1+n)x] + \frac{1}{(1-n)} \cos[(1-n)x] \right]_0^{\pi}$

$$= -\frac{1}{2\pi} \left[\frac{\cos[(n+1)\pi] - 1}{(n+1)} - \frac{\cos[(n-1)\pi] - 1}{(n-1)} \right]$$

20.10

So:

$$a_2 = -\frac{1}{2\pi}\left[\frac{-1-1}{3} - \frac{-1-1}{1}\right] = -\frac{2}{3\pi}$$

$$a_3 = -\frac{1}{2\pi}\left[\frac{1-1}{4} - \frac{1-1}{2}\right] = 0$$

$$a_4 = -\frac{1}{2\pi}\left[\frac{-1-1}{5} - \frac{-1-1}{3}\right] = -\frac{2}{15\pi}$$

$$a_5 = -\frac{1}{2\pi}\left[\frac{1-1}{6} - \frac{1-1}{4}\right] = 0$$

$$a_6 = -\frac{1}{2\pi}\left[\frac{-1-1}{7} - \frac{-1-1}{5}\right] = -\frac{2}{35\pi}$$

$$\boxed{a_{2m+1} = 0 \,, \quad a_{2m} = \frac{-2}{(2m+1)(2m-1)\pi}} \quad m \neq 0$$

What about a_1 ?

From $(*)$: $$a_1 = \frac{1}{2\pi}\int_0^{\pi} \sin(2x)\, dx$$

$$= -\frac{1}{4\pi}\left[\cos(2x)\right]_0^{\pi}$$

$$= -\frac{1}{4\pi}\left[1-1\right] = 0.$$

$$\boxed{a_1 = 0}$$

<u>Next</u>: $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\, f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(nx)\sin x\, dx$$

(Use $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$)

$$\Rightarrow b_n = \frac{1}{2\pi}\int_0^{\pi} \{\cos[(n-1)x] - \cos[(n+1)x]\}\, dx \qquad (**)$$

$n \neq 1$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)}\cancel{\sin[(n-1)x]}^{\,0}_{\,0} - \frac{1}{(n+1)}\cancel{\sin[(n+1)x]}^{\,0}_{\,0}\right]_0^{\pi}$$

$$\boxed{b_n = 0 \qquad n \neq 1}$$

From $(**)$:

$$b_1 = \frac{1}{2\pi}\int_0^{\pi} [1 - \cos(2x)]\, dx$$

$$= \frac{1}{2\pi}\left[x - \cancel{\frac{\sin(2x)}{2}}^{\,0}_{\,0}\right]_0^{\pi} = \frac{1}{2}$$

$$\boxed{b_1 = \frac{1}{2}}$$

So:

$$f(x) = \frac{1}{\pi} - \frac{2}{3\pi}\cos(2x) - \frac{2}{15\pi}\cos(4x) - \frac{2}{35\pi}\cos(6x) \ldots$$
$$+ \frac{1}{2}\sin(x)$$

Note this can be written very compactly as:

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos(2mx)}{(2m-1)(2m+1)}$$

Summary:

1) Learn the formulas (20.6) and (19.2) for a function $f(x)$ with period $p = 2L$.

2) Know how to use trig. formulas in evaluating coefficients. (Not absolutely necessary to learn trig. formulas, but may be helpful to do so!)

~~K §10.1, 10.2, 10.3.~~

K §11.1, 11.2 pp 477–483, 487–489