MATH2100 Lec. 21 (= MATH2011 Lec. 3)

We have for a given $f(x)$, with

$$f(x+2L) = f(x), \quad -\infty < x < \infty$$

that

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$(21.1) \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$n = 1, 2, 3, \ldots$

When does it work? Are there conditions on $f(x)$ for it to work?

There are various theorems....

Suppose that $f(x)$ is <u>sectionally</u> (or <u>piecewise</u>) <u>continuous</u> on the interval $[-L, L]$.

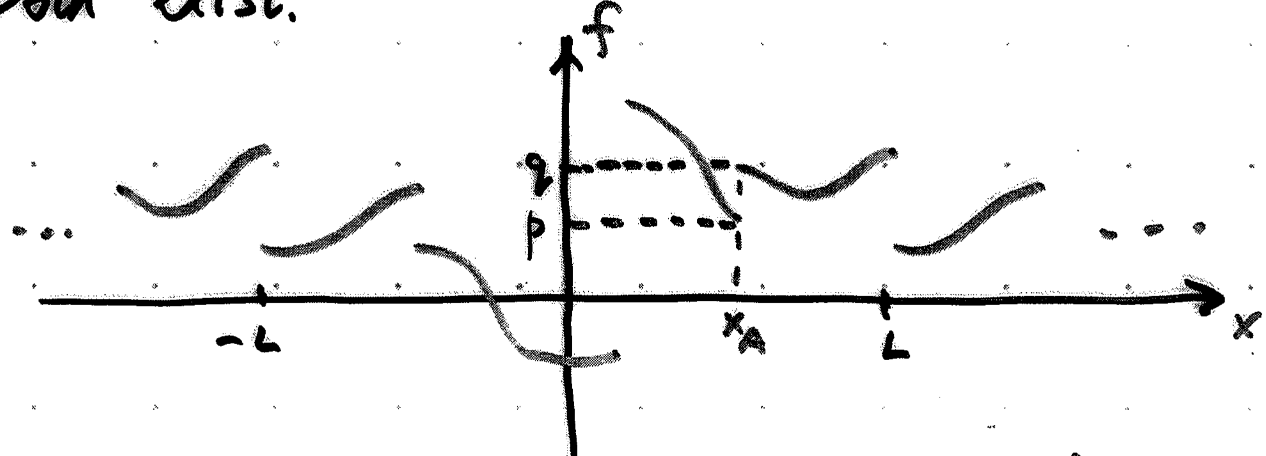
<u>Means</u>: f is made up of a <u>finite</u> number of <u>continuous</u> pieces on this interval, and at each point of discontinuity x_A, the limits

$$\overbrace{\lim_{x \to x_{A+}} f(x)}^{= f(x_A+)} \quad \text{and} \quad \overbrace{\lim_{x \to x_{A-}} f(x)}^{= f(x_A-)}$$

both exist. Also the limits

$$\lim_{x \to L_-} f(x) \quad \text{and} \quad \lim_{x \to -L_+} f(x)$$

both exist.



$f(x_A-) = p \qquad f(x_A+) = q$

<u>Theorem</u>: (K p. ~~535~~ 484) If $f(x)$ is sectionally continuous on the interval $[-L, L]$, and is periodic with period $2L$, then its Fourier series (21.1) converges to the value

$$\tfrac{1}{2}\left[f(x_A+) + f(x_A-)\right]$$

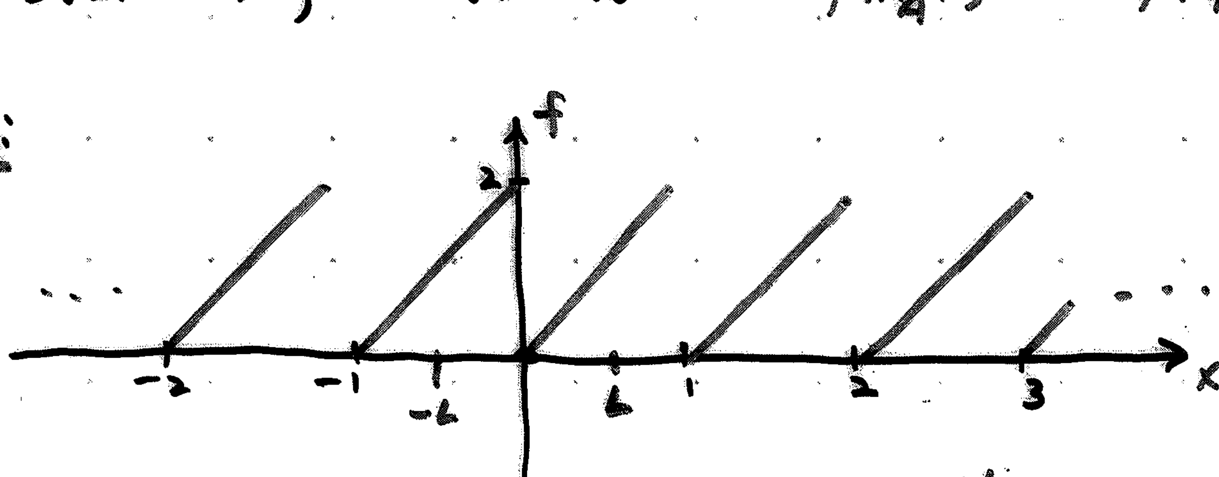
at every point x_A where f has a left and right derivative.

[<u>Means</u>:
$$f'(x_A+) = \lim_{\varepsilon \to 0} \frac{f(x_A+\varepsilon) - f(x_A+)}{\varepsilon} \quad (\varepsilon > 0)$$

$$f'(x_A-) = \lim_{\varepsilon \to 0} \frac{f(x_A-\varepsilon) - f(x_A-)}{\varepsilon} \quad (\varepsilon > 0)$$

both exist, in addition to $f(x_A+)$ and $f(x_A-)$.]

EX:



$L = \tfrac{1}{2}$

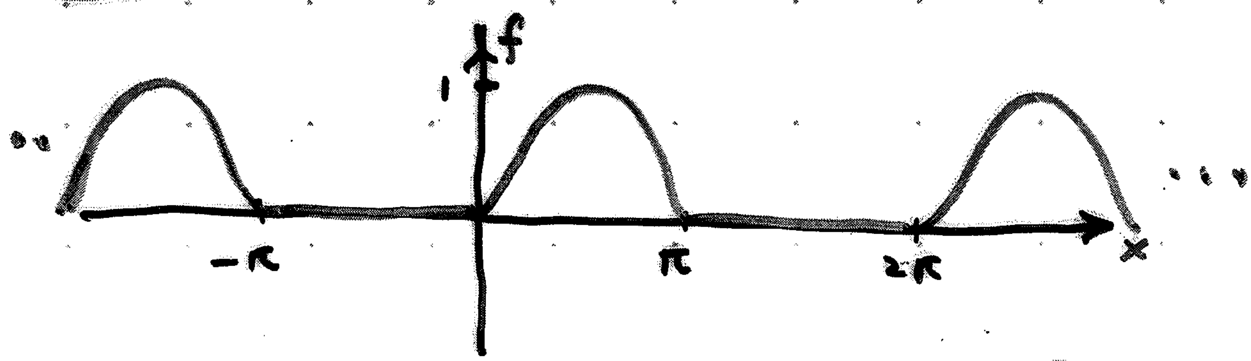
As on p. 20.7,

$$f(x) = 1 - \frac{2}{\pi}\left(\frac{\sin(2\pi x)}{1} + \frac{\sin(4\pi x)}{2} + \cdots\right)$$

This formula is valid in this case except at $x = 0, \pm 1, \pm 2, \ldots$ — the points of discontinuity. At those points, the series converges to $\frac{1}{2}[2+0] = 1$.
[Check: At each of these points, $\sin(2\pi x) = \sin(4\pi x) = \cdots = 0$.]

We could define $f(x) = 1$ at each of these points, then the formula would be correct for all x.

EX:



(See pp. 20.8, 20.12)

In this case, $f(x)$ is everywhere continuous, and has a left and right derivative at every point.

So, for <u>all</u> x, we have

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\left(\frac{1}{3}\cos(2x) + \frac{1}{15}\cos(4x) + \cdots\right)$$

[e.g. $x = \frac{\pi}{2}$: $\quad 1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\left(-\frac{1}{3} + \frac{1}{15} - \frac{1}{35} + \cdots\right)$

$$\Rightarrow \frac{1}{3} - \frac{1}{15} + \frac{1}{35} \cdots = \frac{\pi}{4} - \frac{1}{2}$$]

Also, for <u>all</u> $-\pi < x < 0$

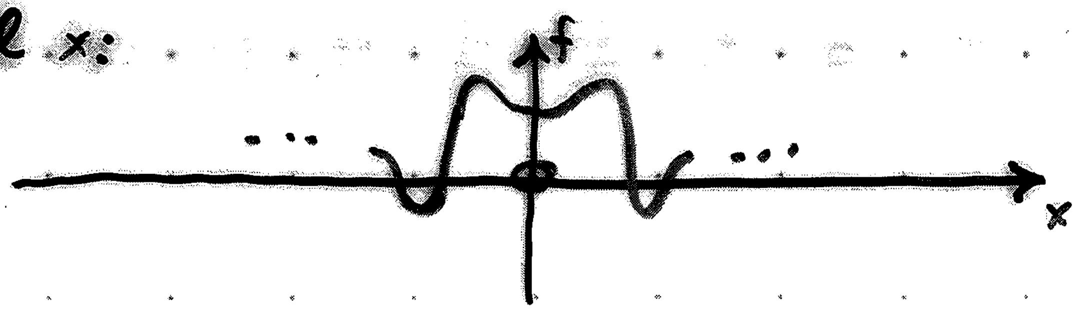
$$0 = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\left(\frac{1}{3}\cos(2x) + \frac{1}{15}\cos(4x) + \cdots\right)$$

$$\Rightarrow \sin(x) = -\frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos(2x) + \frac{1}{15}\cos(4x) + \cdots\right)$$

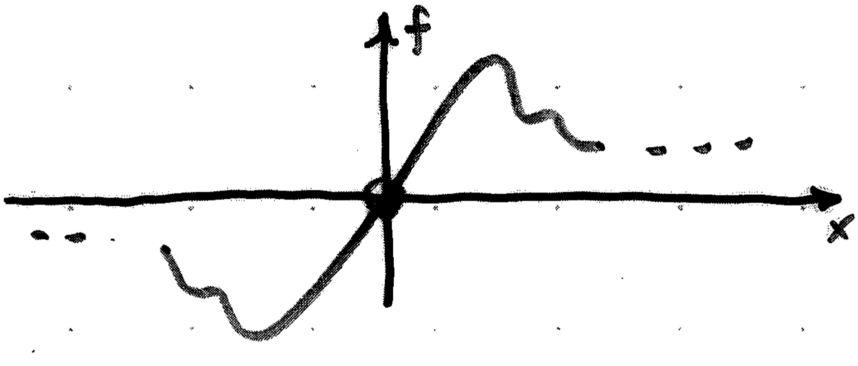
Remarkable!]

Even and Odd Functions

We say $f(x)$ is <u>even</u> if $f(-x) = f(x)$ for all x:



We say $f(x)$ is <u>odd</u> if $f(-x) = -f(x)$ for all x:



We note ~~that~~

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f \text{ even} \\ 0, & f \text{ odd} \end{cases}$$

We note that, if $f(x)$ is <u>even</u> and $g(x)$ is <u>even</u>, then $F(x) = f(x)g(x)$ is <u>even</u>:

$$F(-x) = f(-x)g(-x) = f(x)g(x) = F(x)$$

We note that, if $f(x)$ is <u>odd</u> and $g(x)$ is <u>odd</u>, then $F(x) = f(x)g(x)$ is <u>even</u>:

$$F(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = F(x)$$

We note that, if $f(x)$ is <u>even</u> and $g(x)$ is <u>odd</u>, then $F(x) = f(x)g(x)$ is <u>odd</u>:

$$F(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -F(x)$$

We note that $1, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right) \dots$ are all <u>even</u>, and

$$\sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots$$

are all <u>odd</u>.

Now suppose $f(x)$ is periodic with period $2L$ and is also even:

$$\left.\begin{aligned} f(x+2L) &= f(x) \\ f(-x) &= f(x) \end{aligned}\right\}, \quad -\infty < x < \infty$$

Then

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}} \underbrace{f(x)}_{\text{even}}}_{\text{odd}} dx = 0 \qquad n=1,2,3\ldots$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}} dx = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}} \underbrace{f(x)}_{\text{even}}}_{\text{even}} dx = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx$$

So in this case

Cosine series for even function

$$\boxed{\begin{gathered} f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad (21.2) \\ a_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx \end{gathered}}$$

Similarly if $f(x)$ is periodic with period $2L$ and is also odd:

$$\left.\begin{aligned} f(x+2L) &= f(x) \\ f(-x) &= -f(x) \end{aligned}\right\} , \quad -\infty < x < \infty$$

then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{\text{odd}}\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}} \underbrace{f(x)}_{\text{odd}}}_{\text{odd}}\,dx = 0$$

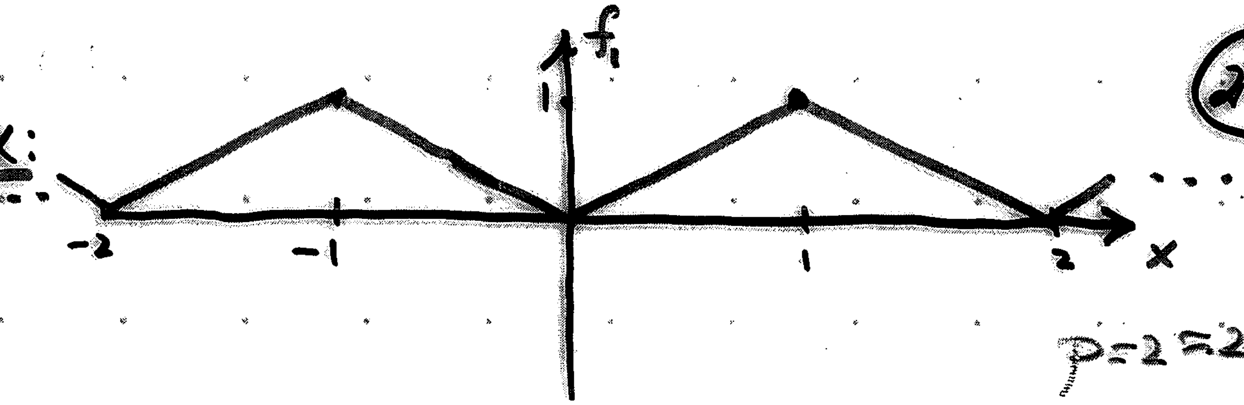
$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}} \underbrace{f(x)}_{\text{odd}}}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right) f(x)\,dx$$

So in this case

Sine series for odd function

$$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \\ b_n &= \frac{2}{L}\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right) f(x)\,dx \end{aligned}} \quad (21.3)$$

EX:



$p = 2 = 2L$

$$f_1(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x, & -1 < x < 0 \end{cases}, \quad f_1(x+2) = f_1(x), \quad -\infty < x < \infty.$$

[**OR** $f_1(x) = |x| \quad -1 < x \le 1, \quad f_1(x+2) = f_1(x) \quad -\infty < x < \infty$]

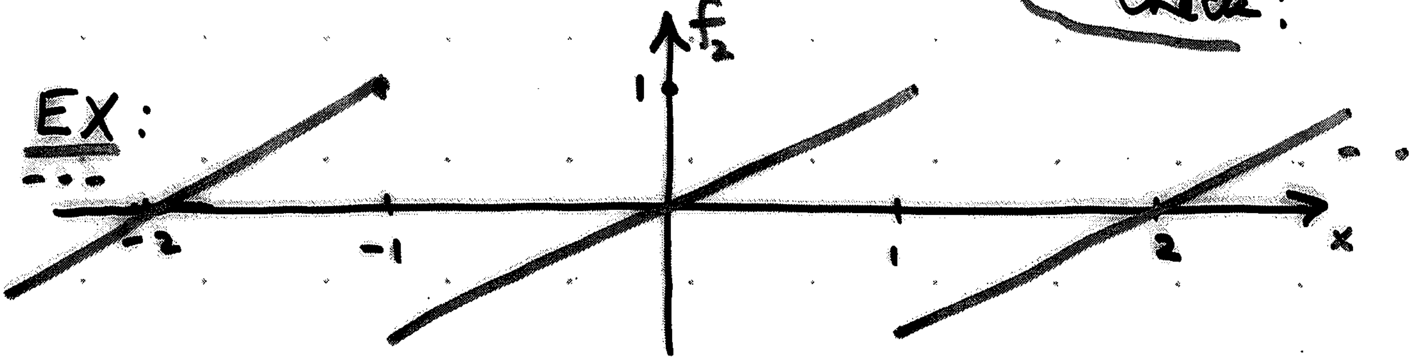
Even function with period $2L = 2$

$$f_1(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_0 = \int_0^1 x\,dx = \frac{1}{2} \qquad a_n = 2\int_0^1 x \cos(n\pi x)\,dx = \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right]$$

check!

EX:



$f_2(x) = x \quad -1 < x < 1 \qquad f_2(x+2) = f_2(x) \quad -\infty < x < \infty$

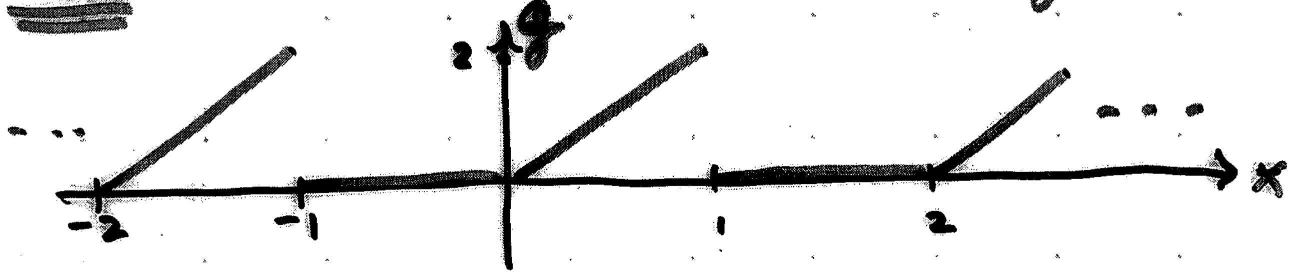
Odd function with period $2L = 2$

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = 2\int_0^1 x \sin(n\pi x)\,dx = -\left(\frac{2}{n\pi}\right)(-1)^n$$

Check!

Note: The Fourier series for $g(x) = f_1(x) + f_2(x)$



is just the sum of the cosine series for f_1 and the sine series for f_2.

[Any periodic function with period $2L$ can be written as the sum of an even periodic function f_1 with period $2L$, and an odd periodic function f_2 with period $2L$.

– just set
$$f_1(x) = \tfrac{1}{2}[g(x) + g(-x)]$$
$$f_2(x) = \tfrac{1}{2}[g(x) - g(-x)]$$

See $f_1(-x) = f_1(x)$, $f_2(-x) = -f_2(x)$
$$g(x) = f_1(x) + f_2(x)$$

The Fourier series for $g(x)$ is the sum of the cosine series for $f_1(x)$ and the sine series for $f_2(x)$.]

Summary:

1) Understand what the Theorem on p. 21.3 is saying. (No proof.)

2) Understand ideas of even and odd functions, their products and integrals.

3) Learn formulas for cosine and sine series for even/odd functions

~~K pp. 534, 535, 541, 542, 543~~

K pp 484, 485, 490–495