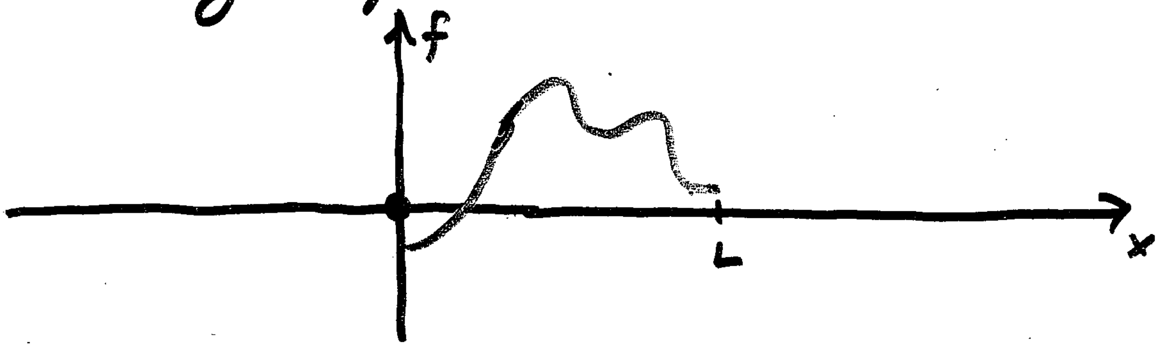
Lec. 22 MATH2100 (= Lec. 4 MATH2011)

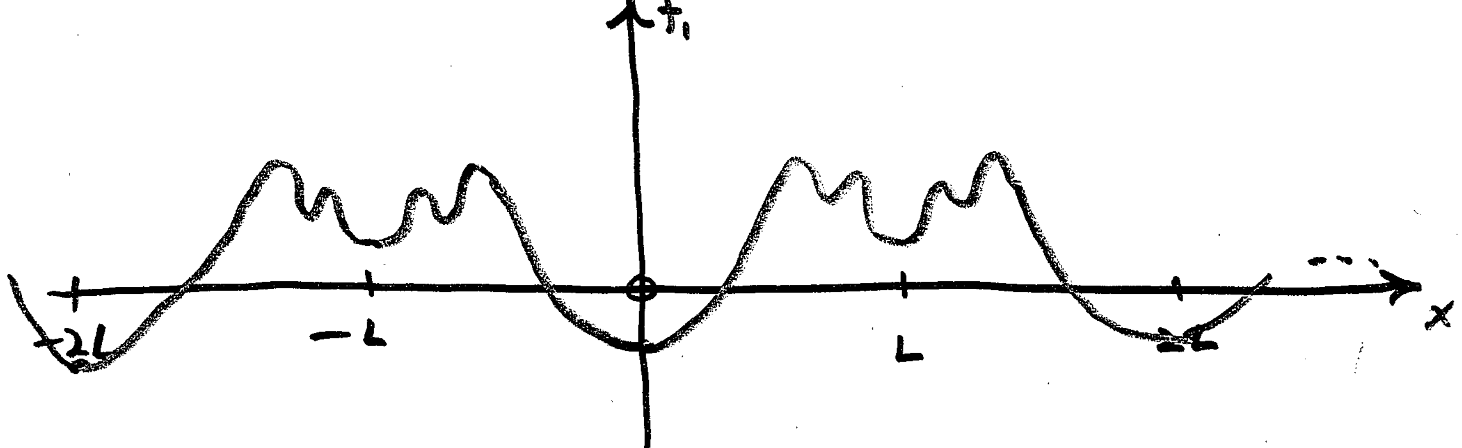
Suppose we are given a function $f(x)$ that is <u>only defined for</u> $0 \leq x \leq L$:



May still be useful to have a Fourier Series decomposition.

Two obvious ways to proceed:-

1) **Etend** to <u>even periodic function</u>:



$f_1(x) = f(x), \quad 0 \leq x \leq L$

$f_1(x+2L) = f_1(x), \quad f_1(-x) = f_1(x), \quad -\infty < x < \infty$

Then

$$f_1(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad -\infty < x < \infty$$

$$\Rightarrow \quad \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L} \quad (22.1)$$

(with usual conditions about jump discontinuities).

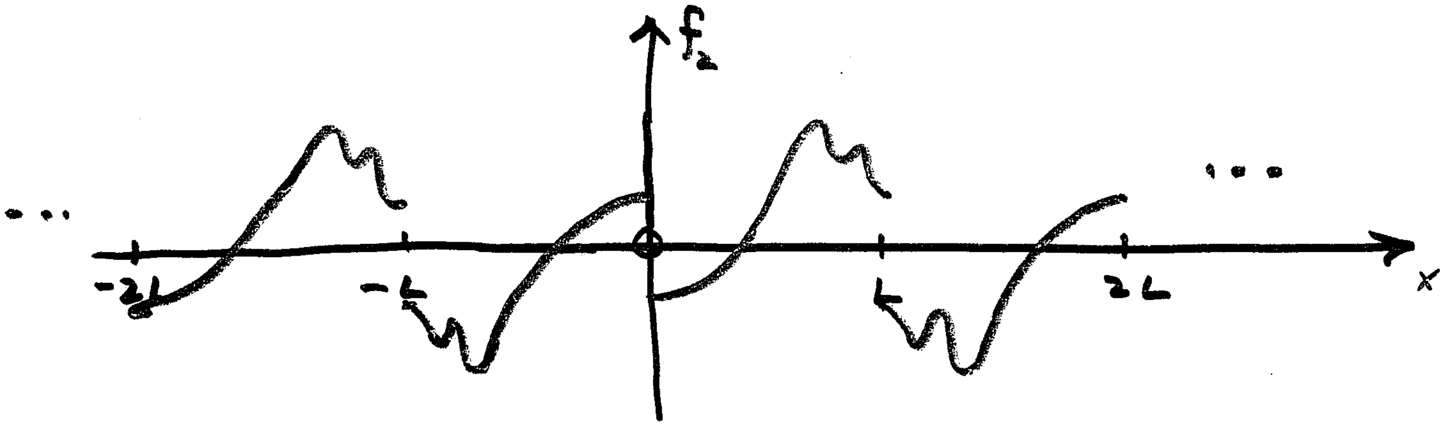
Here

$$\boxed{\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f_1(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f_1(x)\,dx = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx \end{aligned}} \quad (22.2)$$

We call this the <u>half-range cosine series</u> for $f(x)$, $0 \leq x \leq L$.

(It is also the cosine series for the even periodic extension $f_1(x)$, $-\infty < x < \infty$.)

2) <u>Extend to</u> <u>odd periodic function:</u>



$$f_2(x) = f(x) \,, \quad 0 \le x \le L$$

$$f_2(x+2L) = f_2(x) \,, \quad f_2(-x) = -f_2(x) \,,$$
$$-\infty < x < \infty$$

Then
$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \,, \quad -\infty < x < \infty$$

$$\Rightarrow \quad \boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \,, \quad 0 \le x \le L} \tag{22.3}$$

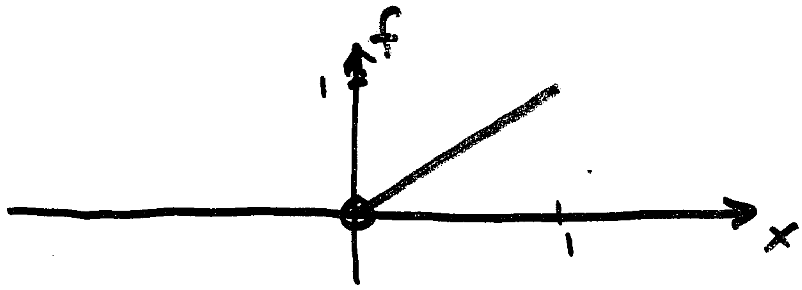
(with usual conditions about jumps)

Here
$$\boxed{b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f_2(x)\,dx = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx} \tag{22.4}$$

We call this the <u>half-range sine series</u> for $f(x)$, $0 \le x \le L$.

(It is also the sine series for the odd periodic extension $f_2(x)$, $-\infty < x < \infty$.)

EX:



$$f(x) = x \quad , \quad 0 \le x \le 1 \qquad (L=1)$$

<u>Half-range cosine series:</u>

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x), \qquad 0 \le x \le 1$$

$$a_0 = \frac{1}{1}\int_0^1 x\,dx = \frac{1}{2}$$

$$a_n = \frac{2}{1}\int_0^1 x \cos(n\pi x)\,dx = \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right]$$

So

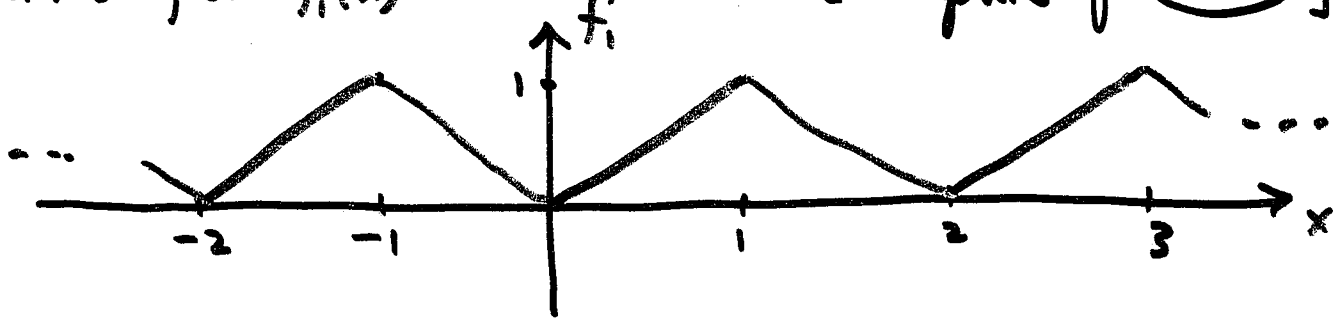
$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos(\pi x)}{1} + \frac{\cos(3\pi x)}{9} + \frac{\cos(5\pi x)}{25} \dots\right.$$

$\parallel$

x

$$0 \le x \le 1$$

See this is also the Fourier cosine series for $f_1(x)$: [compare p 21.10]



<u>Half-range sine series</u>:

$$\underset{\substack{\| \\ x}}{f(x)} = \sum_{n=1}^{\infty} b_n \sin(n\pi x), \qquad 0 \le x \overset{<1}{\le 1}$$

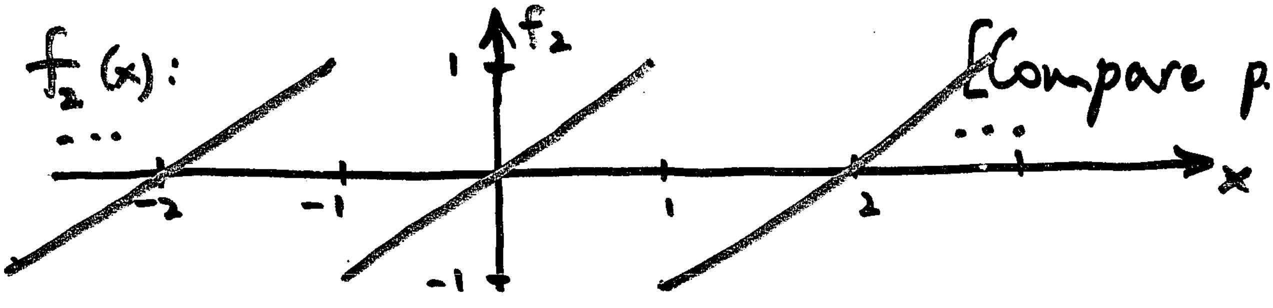
$$b_n = \frac{2}{1}\int_0^1 x \sin(n\pi x)\,dx = -\frac{2}{n\pi}(-1)^n$$

So $$f(x) = \frac{2}{\pi}\left[\frac{\sin(\pi x)}{1} - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} \cdots\right]$$

$$0 \le x \le 1$$

[Actually, at $x=1$, series converges to 0, not 1.

See this is also the Fourier sine series for $f_2(x)$: [Compare p 21.11]



Differentiating a Fourier Series

Suppose $f(x)$ is periodic with period $2L$,

and

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}} \tag{22.5}$$

Suppose also that $f'(x)$ is (at least) sectionally continuous.

Now $f'(x)$ is also periodic, with period $2L$:

$$f'(x+2L) = \lim_{\varepsilon \to 0} \frac{\overset{f(x+\varepsilon)}{\overset{\shortparallel}{f(x+2L+\varepsilon)}} - \overset{f(x)}{\overset{\shortparallel}{f(x+2L)}}}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

$$= f'(x)$$

So

$$\boxed{f'(x) = A_0 + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right\}} \tag{22.6}$$

What is the relation between (22.5) and (22.6)?
We know

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{2L}\Big[f(x)\Big]_{-L}^{L} = 0$$

as $f(L) = f(-L)$

$$A_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f'(x)}_{u'} \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{v} dx$$

$$= \frac{1}{L}\left\{ \Big[\underbrace{f(x)}_{u} \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{v}\Big]_{-L}^{L} + \left(\frac{n\pi}{L}\right) \underbrace{\int_{-L}^{L} \underbrace{f(x)}_{u} \sin\left(\frac{n\pi x}{L}\right) dx}_{-v'} \right\}$$

$$= 0 + \left(\frac{n\pi}{L}\right) \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \left(\frac{n\pi}{L}\right) b_n$$

Similarly

$$B_n = \frac{1}{L}\int_{-L}^{L} f'(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\left\{ \left[f(x)\sin\left(\frac{n\pi x}{L}\right) \right]_{-L}^{L} - \left(\frac{n\pi}{L}\right) \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \right\}$$

$$= 0 - \left(\frac{n\pi}{L}\right) \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$= -\left(\frac{n\pi}{L}\right) a_n$$

We see then ~~that~~ (22.6) is just what we get by differentiating (22.5) term by term!

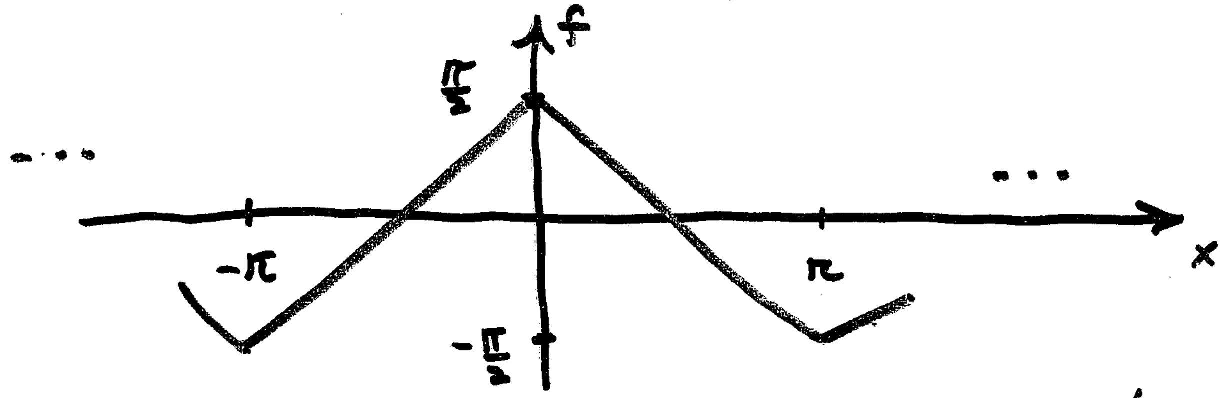
(22.5): $$f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

$$\Rightarrow \quad f'(x) = \underbrace{0}_{A_0} + \sum_{n=1}^{\infty} \left\{ \underbrace{-\left(\frac{n\pi}{L}\right) a_n}_{B_n} \sin\left(\frac{n\pi x}{L}\right) + \underbrace{\left(\frac{n\pi}{L}\right) b_n}_{A_n} \cos\left(\frac{n\pi x}{L}\right) \right\}$$

Basically: can differentiate a F.S. term by term, so long as result is the F.S. for a sectionally continuous function.

Ex: $f(x) = \begin{cases} x + \frac{\pi}{2}, & -\pi < x < 0 \\ -x + \frac{\pi}{2}, & 0 < x < \pi \end{cases}$

$$f(x + 2\pi) = f(x), \qquad -\infty < x < \infty$$



$p = 2L = 2\pi$

Even function: $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_0^\pi \left(-x + \frac{\pi}{2}\right)dx = \frac{1}{\pi}\left[-\frac{1}{2}x^2 + \frac{\pi x}{2}\right]_0^\pi = 0$$

$$a_n = \frac{2}{\pi}\int_0^\pi \left(-x + \frac{\pi}{2}\right)\cos(nx)\,dx$$ (see p. 20.6)

$$= \frac{2}{\pi}\left[-\underbrace{\cancel{\frac{x}{n}\sin(nx)}}_{0} - \frac{1}{n^2}\cos(nx) + \underbrace{\cancel{\frac{\pi}{2n}\sin(nx)}}_{0}\right]_0^\pi$$

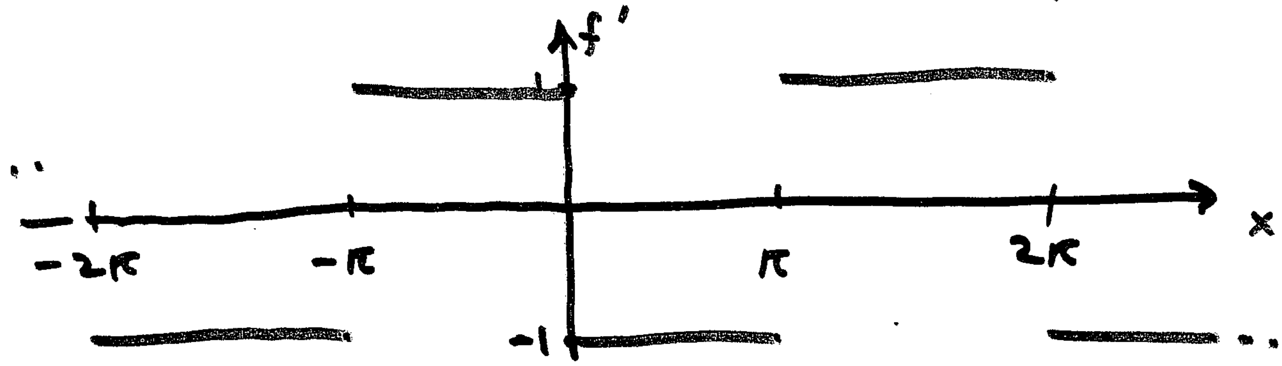
$$= -\frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} \frac{4}{\pi(2m+1)^2}, & n = (2m+1) \text{ odd} \\ 0, & n = 2m \quad \text{even} \end{cases}$$

$$f(x) = \frac{4}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \cdots\right)$$

(22.7)

In this case $f'(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases}$

$$f'(x+2\pi) = f'(x) \quad , \quad -\infty < x < \infty$$



($f(x)$ is conts., $f'(x)$ is sectionally conts.)

Note also: $f(x)$ is even, $f'(x)$ is odd.

From (22.7):

$$f'(x) = \frac{4}{\pi}\left(-\sin(x) - \frac{3}{9}\sin(3x) - \frac{5}{25}\sin(5x) \ldots\right)$$

$$= -\frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) \ldots\right)$$

Check: $f'(x)$ is odd, with period 2π, so

$$f'(x) = \sum_{n=1}^{\infty} B_n \sin(nx) \quad , \quad B_n = \frac{2}{\pi}\int_0^{\pi} (-1)\sin(nx)\,dx$$

$$\Rightarrow \quad B_n = \frac{2}{n\pi}\left[\cos(nx)\right]_0^{\pi} = \frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} -\dfrac{4}{(2m+1)\pi}, & n = (2m+1) \quad \text{(odd)} \\ 0, & n = 2m \quad \text{(even)} \end{cases}$$

It <u>works!</u>

<u>Note</u>: In general, if $f(x)$ is <u>even</u>, then $f'(x)$ is <u>odd</u>:

$$f'(-x) = \lim_{\varepsilon \to 0} \frac{\overset{f(x-\varepsilon)}{\overset{\shortparallel}{f(-x+\varepsilon)}} - \overset{f(x)}{\overset{\shortparallel}{f(-x)}}}{+\varepsilon}$$

$$= -\lim_{\varepsilon \to 0} \frac{f(x) - f(x-\varepsilon)}{\varepsilon}$$

$$= -f'(x)$$

And if $f(x)$ is <u>odd</u>, then $f'(x)$ is <u>even</u>:

$$f'(-x) = \lim_{\varepsilon \to 0} \frac{\overset{-f(x-\varepsilon)}{\overset{\shortparallel}{f(-x+\varepsilon)}} - \overset{-f(x)}{\overset{\shortparallel}{f(-x)}}}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{f(x) - f(x-\varepsilon)}{\varepsilon}$$

$$= f'(x)$$

Summary:

1) Understand how to obtain Fourier half-range sine and cosine series for a function $f(x)$ that is only defined for $0 \le x \le L$.

– learn formulas for coefficients.

2) Understand how to differentiate a Fourier series term by term, and in general terms when it will work.

~~K 544 – 6~~

K 493 – 5