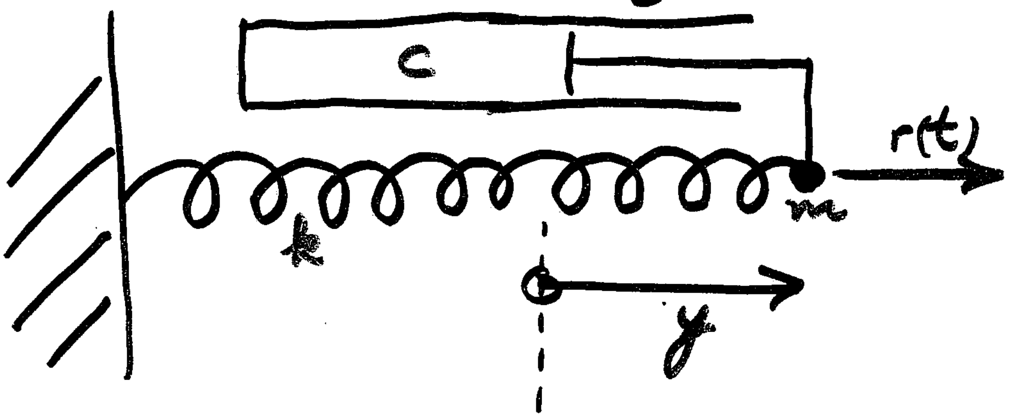
Lec. 23 MATH2100 (= Lec. 5 MATH2011)

An application: Forced oscillations

Consider forced mass-spring-damper system:



$$m y''(t) = \underbrace{-c y'(t)}_{\text{damping force}} \underbrace{- k y(t)}_{\text{restoring force}} + \underbrace{r(t)}_{\text{applied force}} \quad (23.1)$$

Suppose r(t) is periodic with period 2L, and has been applied for a long time — all effects of ICs have died away, so system is in a steady-state.

Then y(t) will be periodic with period 2L, and same will be true for y'(t), y''(t).

We can plug the Fourier series for y''(t), y'(t), y(t), and r(t) in (23.1) and equate coefficients:

cf. K. p.550

EX: (~~K. p.551~~)

$$y''(t) + \left(\frac{1}{50}\right) y'(t) + 25\,y(t) = r(t) \quad (23.2)$$

$$r(t) = \begin{cases} t + \frac{\pi}{2}, & -\pi < t < 0 \\ -t + \frac{\pi}{2}, & 0 < t < \pi \end{cases}$$

$$r(t+2\pi) = r(t), \quad -\infty < t < \infty$$

Then (see p. 22.9):

$$r(t) = \frac{4}{\pi}\left(\cos(t) + \frac{1}{9}\cos(3t) + \frac{1}{25}\cos(5t) + \cdots\right)$$

Suppose
$$y(t) = a_0 + \sum_{n=1}^{\infty}\{a_n \cos(nt) + b_n \sin(nt)\}$$

$$\Rightarrow \quad y'(t) = 0 + \sum_{n=1}^{\infty}\{-n a_n \sin(nt) + n b_n \cos(nt)\}$$

$$\Rightarrow \quad y''(t) = 0 + \sum_{n=1}^{\infty}\{-n^2 a_n \cos(nt) - n^2 b_n \sin(nt)\}$$

Now substitute in (23.2):

$$\left\{ -a_1 \cos(t) - b_1 \sin(t) - 4a_2 \cos(2t) - 4b_2 \sin(2t) - 9 \ldots \right\}$$

$$+\left(\frac{1}{50}\right)\left\{ -a_1 \sin(t) + b_1 \cos(t) - 2a_2 \sin(2t) + 2b_2 \cos(2t) - 3 \ldots \right\}$$

$$+25\left\{ a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + \cdots \right.$$

$$= \frac{4}{\pi}\left\{ \cos(t) + \frac{1}{9}\cos(3t) + \frac{1}{25}\cos(5t) + \cdots \right\}$$

Equating coefficients:

$$25a_0 = 0 \qquad a_0 = 0$$

$$\left.\begin{array}{l} -a_1 + \frac{1}{50}b_1 + 25a_1 = \frac{4}{\pi} \\ -b_1 - \frac{1}{50}a_1 + 25b_1 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} a_1 = 0.0530\ldots \\ b_1 = 4.420\ldots \times 10^{-5} \end{array}$$

etc.

In this way we determine the solution of the ODE in terms of its Fourier series:

$$y(t) = (0.0530\ldots)\cos(t) + (4.420\ldots \times 10^{-5})\sin(t) + \cdots.$$

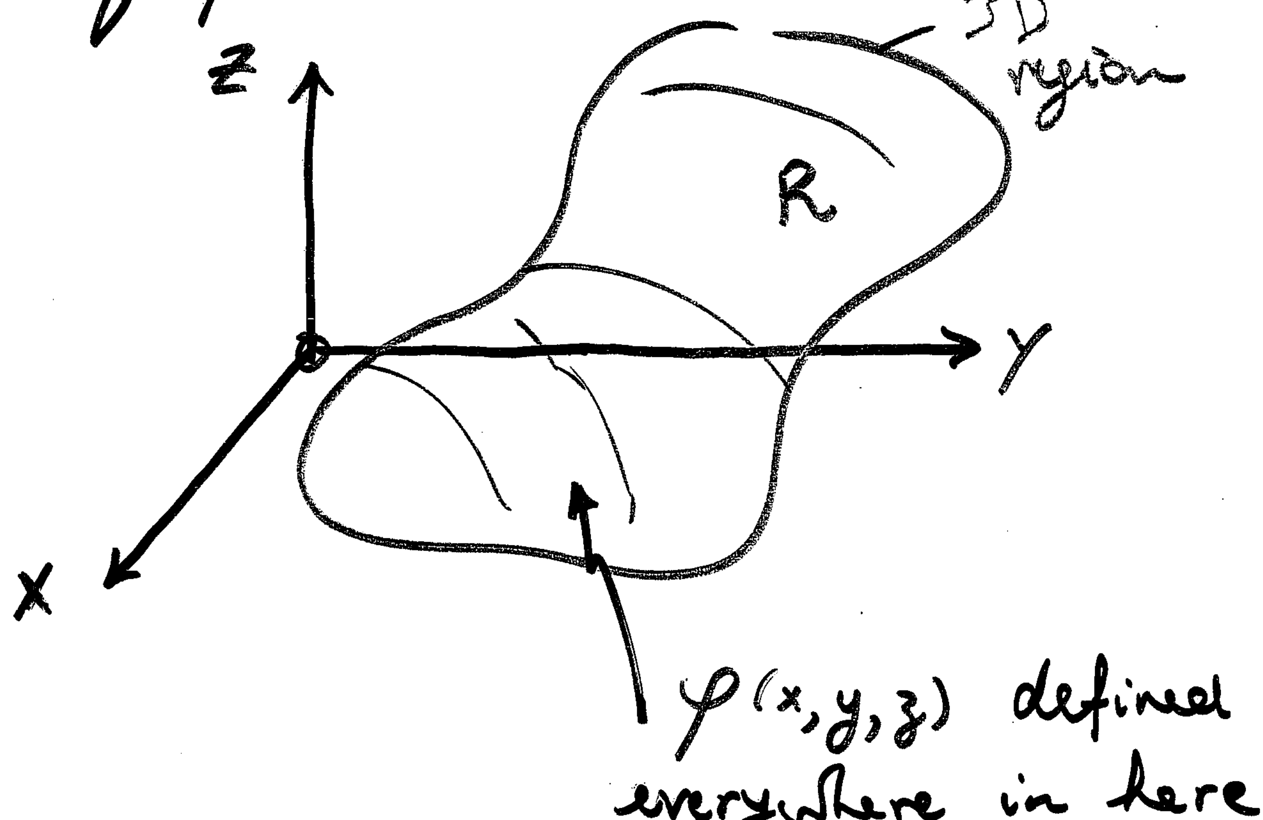
Can get a formula for a_n, b_n, but in practice would just calculate as many terms as needed to get a suitably good <u>approximate solution</u>.

Look at K pp ~~551-2~~ 500-501. Note how the component of $r(t)$ at the frequency of the undamped oscillator ($\bar{\omega} = \sqrt{\frac{k}{m}} = 5$ here) gives the dominant contribution to the solution.

We will look at further applications of Fourier series later (with PDEs).

Scalar fields

A scalar field is a function of position, defined throughout some region R of space.



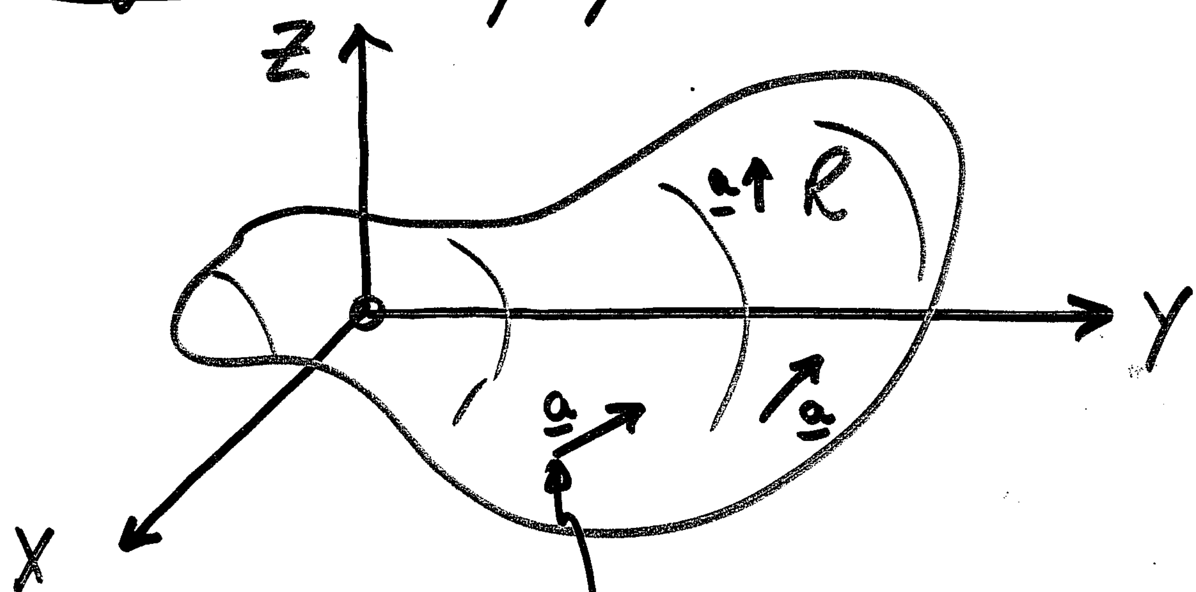
EX: $\varphi = 6xyz + 3\cos(yz)e^{x^3}$

R: $0 < x \leq 1, \quad -\infty < y < \infty, \quad z > 0$

EX: Temperature at each point in this room.

Vector fields

A *vector field* is a *vector* which is a *function of position*, defined throughout some *region* R of space.



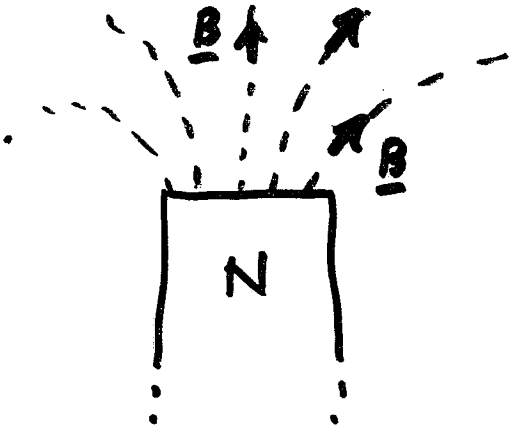
$\underset{\sim}{a}(x,y,z)$ defined everywhere in here.

EX: $$\underset{\sim}{a} = 6xy\,\underset{\sim}{i} + 3e^{yz}\,\underset{\sim}{j} + 10\cos(xyz)\,\underset{\sim}{k}$$

R: $-\infty < x < 0$, $2 < y < 4$, $-\infty < z < \infty$

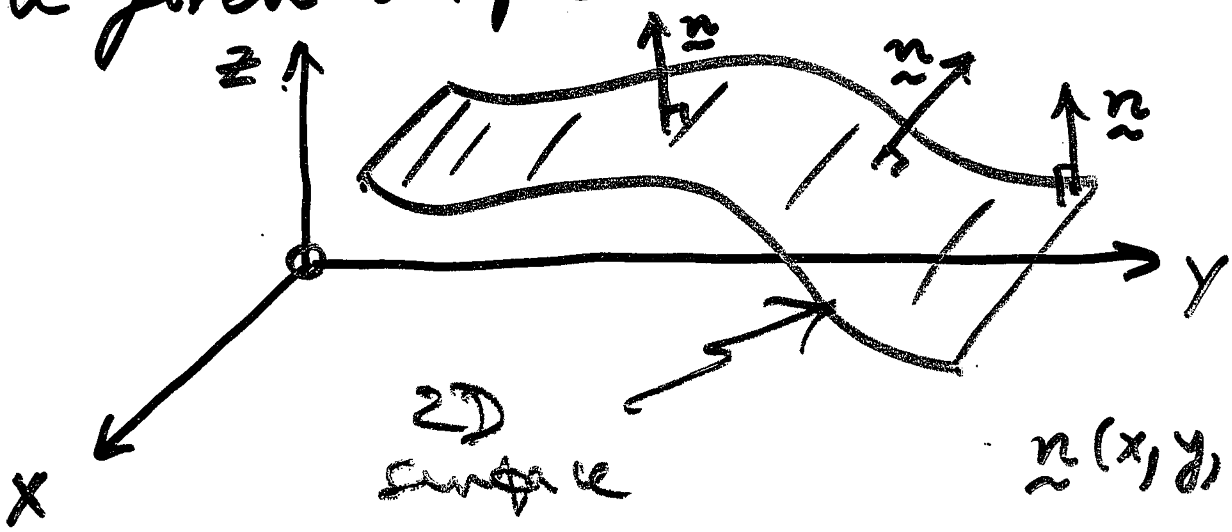
EX: Velocity of water at each point in Brisbane River, at a given time. $\underset{\sim}{v}(x,y,z)$

Ex: Magnetic field outside a bar magnet:



$\underline{B}(x, y, z)$

Ex: Unit vector perpendicular to a given surface:



$\underset{\sim}{n}(x, y, z)$

Scalar and vector fields often depend also on the <u>time</u> t:

$\varphi(x, y, z, t)$, $\underset{\sim}{B}(x, y, y, t)$

EXS: Temperature at each point in this room, at time t: $T(x,y,z,t)$

Velocity of water at each point in Brisbane River, at time t:

$$\underset{\sim}{v}(x,y,z,t)$$

We may be interested in various derivatives:

$$\frac{\partial \varphi(x,y,z,t)}{\partial t}, \quad \frac{\partial^2 \varphi(x,y,z)}{\partial y\, \partial z}, \ldots$$

EX: $\varphi(x,y,z,t) = 6xyz + e^{3x^3} + y\cos(t)$

$$\Rightarrow \frac{\partial \varphi}{\partial y} = 6xz + \cos(t)$$

$$\Rightarrow \frac{\partial^2 \varphi}{\partial x\, \partial y} = 6z, \quad \frac{\partial^2 \varphi}{\partial t\, \partial y} = -\sin(t) \quad \text{etc.}$$

Similarly for vector fields:

EX: $\underset{\sim}{a}(x,y,z,t) = \sin(x^2y)\,\underset{\sim}{i} + e^{xyz}\,\underset{\sim}{j} + tx^4y^3z^2\,\underset{\sim}{k}$

$$\Rightarrow \quad \frac{\partial \underset{\sim}{a}}{\partial x} = 2xy\cos(x^2y)\,\underset{\sim}{i} + yz\,e^{xyz}\,\underset{\sim}{j} + 4tx^3y^3z^2\,\underset{\sim}{k}$$

$$\Rightarrow \quad \frac{\partial^2 \underset{\sim}{a}}{\partial t\,\partial x} = 4x^3y^3z^2\,\underset{\sim}{k} \qquad \text{etc.}$$

We can think of $\frac{\partial \varphi}{\partial y}$ for example, as the result of applying the <u>differential operator</u> $\frac{\partial}{\partial y}$ to φ.

Similarly, applying the <u>second-order differential operator</u> $\frac{\partial^2}{\partial x\,\partial z}$ to φ gives

$$\frac{\partial^2 \varphi}{\partial x\,\partial z}$$

A particular combination of vectors and differential operators appears in many applications:

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$$

[Called **del** : a <u>vector</u> <u>operator</u>]

<u>Action:</u> Applying $\underset{\sim}{\nabla}$ to a scalar field $\varphi(x,y,z)$ we get

$$\underset{\sim}{\nabla}\varphi = \frac{\partial \varphi}{\partial x}\underset{\sim}{i} + \frac{\partial \varphi}{\partial y}\underset{\sim}{j} + \frac{\partial \varphi}{\partial z}\underset{\sim}{k}$$

$$= \underset{\sim}{i}\frac{\partial \varphi}{\partial x} + \underset{\sim}{j}\frac{\partial \varphi}{\partial y} + \underset{\sim}{k}\frac{\partial \varphi}{\partial z}$$

– read as **del phi** or **grad phi**

– the <u>gradient</u> of phi.

<u>EX:</u> $\varphi(x,y,z) = 6xy^2z^3$

$$\underset{\sim}{\nabla}\varphi(x,y,z) = 6y^2z^3\underset{\sim}{i} + 12xyz^3\underset{\sim}{j} + 18xy^2z^2\underset{\sim}{k}$$

— a vector field!

We can also act with the vector operator $\underset{\sim}{\nabla}$ on a given vector field $\underset{\sim}{a}(x, y, z)$ in two ways:

If $\underset{\sim}{a}(x,y,z) = a_1(x,y,z)\underset{\sim}{i} + a_2(x,y,z)\underset{\sim}{j} + a_3(x,y,z)\underset{\sim}{k}$

we can form the <u>scalar</u> product

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{a} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right) \cdot \left(a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}\right)$$

$$= \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

– the <u>divergence</u> of $\underset{\sim}{a}$, or del dot $\underset{\sim}{a}$, or div $\underset{\sim}{a}$ — a scalar field

<u>AND</u>

we can also form the <u>vector</u> product

$$\underset{\sim}{\nabla} \times \underset{\sim}{a} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right) \times \left(a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}\right)$$

$$= \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)\underset{\sim}{i} + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\underset{\sim}{j} + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\underset{\sim}{k}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

– read as del cross $\underset{\sim}{a}$ or curl $\underset{\sim}{a}$

– a vector field.

Summary:

1) Understand how to use Fourier Series to find steady-state periodic solution of a mechanical or electrical oscillator with periodic forcing.

2) Concepts of scalar and vector fields.

3) Know the operator $\underset{\sim}{\nabla}$ and how to form

$\underset{\sim}{\nabla}\varphi$, $\underset{\sim}{\nabla}\cdot\underset{\sim}{a}$, $\underset{\sim}{\nabla}\times\underset{\sim}{a}$

~~K §10.6, pp. 423–4, 427, 446–7, 453, 457~~

K §11.5, 403–6, 410–413, 414–415