Lec. 24 MATH2100 (= Lec. 6 MATH2011)

Given a scalar field $\varphi(x, y, z)$ and a point $P: (x_0, y_0, z_0)$, the equation

$$\varphi(x, y, z) = \varphi(x_0, y_0, z_0)$$

defines a 2-D surface S passing through P.

On this surface S, $\varphi(x, y, z)$ has the constant value $\varphi(x_0, y_0, z_0)$. We call S a <u>level surface</u> for the field φ. (<u>The</u> level surface that passes through the point P.)

$\nabla\varphi$ evaluated at P is a vector $\perp$r to the level surface through P.

direction of $\nabla\varphi$ = direction at P of <u>greatest rate of increase</u> of φ with changing position.

magnitude $|\nabla\varphi|$ = size of greatest rate of increase.

EX1: $\varphi(x,y,z) = x^2+y^2+z^2 \qquad P: (1,2,3)$

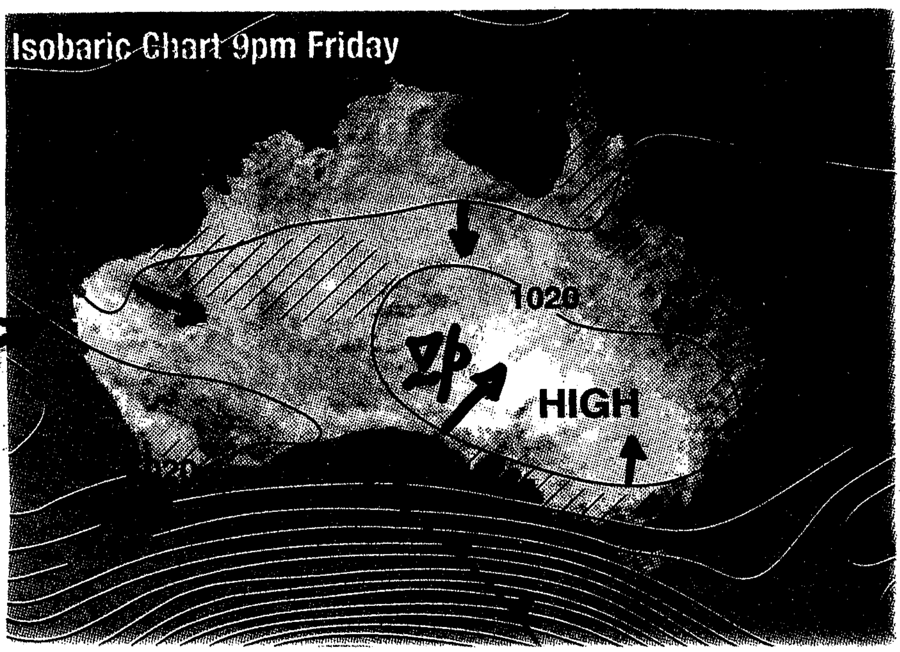
$\varphi(1,2,3) = 1+4+9 = 14$

$\varphi(x,y,z) = 14$ is sphere through P.

$x^2+y^2+z^2 = 14$. Radius $\sqrt{14}$

$\nabla\varphi = 2x\,\underset{\sim}{i} + 2y\,\underset{\sim}{j} + 2z\,\underset{\sim}{k}$

EX2:



line of constant pressure p
(isobar)

$\underset{\sim}{\nabla}\cdot\underset{\sim}{a}$ $\underset{\sim}{\nabla}\times\underset{\sim}{a}$

$\mathrm{div}\,\underset{\sim}{a}$ and $\mathrm{curl}\,\underset{\sim}{a}$ give information about the behaviour of $\underset{\sim}{a}$ as position is changed:-

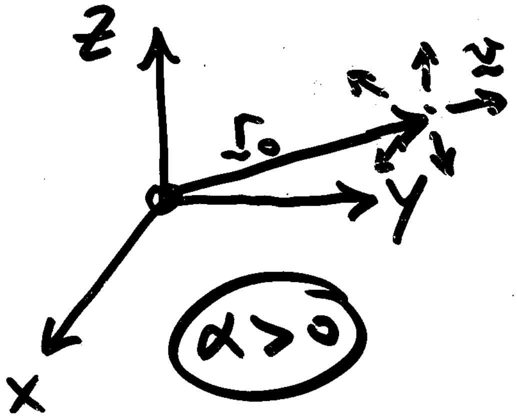
Let $\underset{\sim}{a} = \underset{\sim}{v}(x,y,z)$ be velocity field in moving fluid.

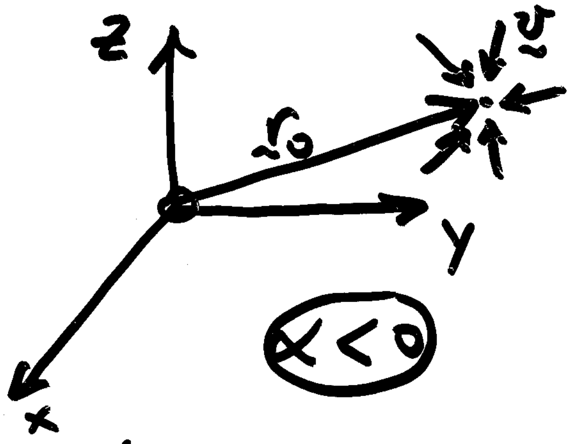
$\mathrm{div}\,\underset{\sim}{v}$ is measure of tendency of flow to be <u>diverging</u> at any point.

<u>EX:</u> $\underset{\sim}{v} = \alpha\left[(x-x_0)\underset{\sim}{i} + (y-y_0)\underset{\sim}{j} + (z-z_0)\underset{\sim}{k}\right]$

$$\Rightarrow \mathrm{div}\,\underset{\sim}{v} = \frac{\partial}{\partial x}[\alpha(x-x_0)] + \frac{\partial}{\partial y}[\alpha(y-y_0)] + \frac{\partial}{\partial z}[\alpha(z-z_0)]$$

$$= \alpha + \alpha + \alpha = 3\alpha$$





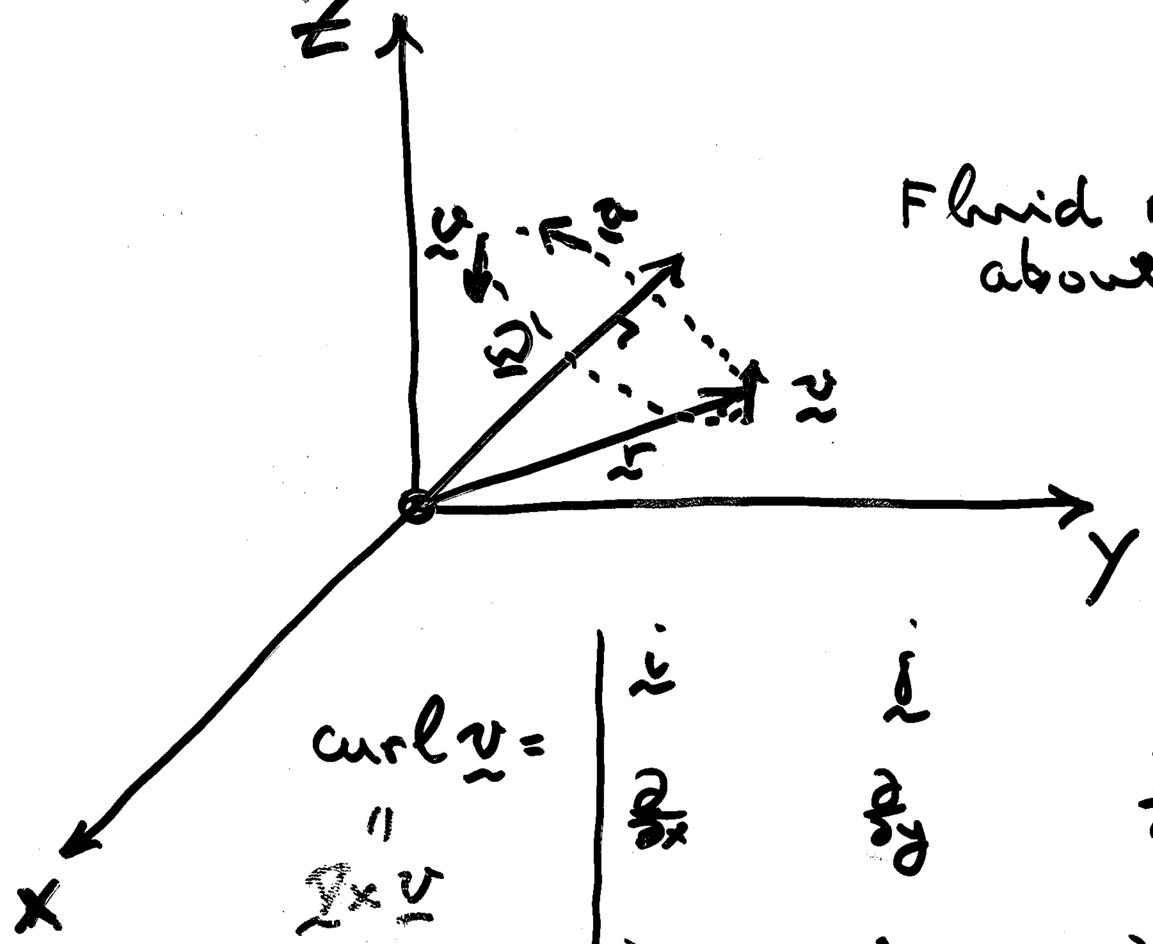
$$\underset{\sim}{r}_0 = x_0\underset{\sim}{i} + y_0\underset{\sim}{j} + z_0\underset{\sim}{k}$$

curl $\underset{\sim}{v}$ is measure of tendency of flow to be <u>rotating</u> at any point.

<u>EX:</u> $\underset{\sim}{v} = \underset{\sim}{\omega} \times \underset{\sim}{r}$

$$\left\{\begin{array}{l} \underset{\sim}{\omega} = \omega_1 \underset{\sim}{i} + \omega_2 \underset{\sim}{j} + \omega_3 \underset{\sim}{k} \\ \text{(constant)} \\ \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \\ \text{(position vector)} \end{array}\right.$$

$$\Leftarrow \quad \underset{\sim}{v} = (\omega_2 z - \omega_3 y)\underset{\sim}{i} + (\omega_3 x - \omega_1 z)\underset{\sim}{j} + (\omega_1 y - \omega_2 x)\underset{\sim}{k}$$



Fluid rotates about $\underset{\sim}{\omega}$

$$\operatorname{curl}\underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$\operatorname{curl}\underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{v}$$

$$= \underset{\sim}{i}(\omega_1 + \omega_1) - \underset{\sim}{j}(-\omega_2 - \omega_2) + \underset{\sim}{k}(\omega_3 + \omega_3)$$

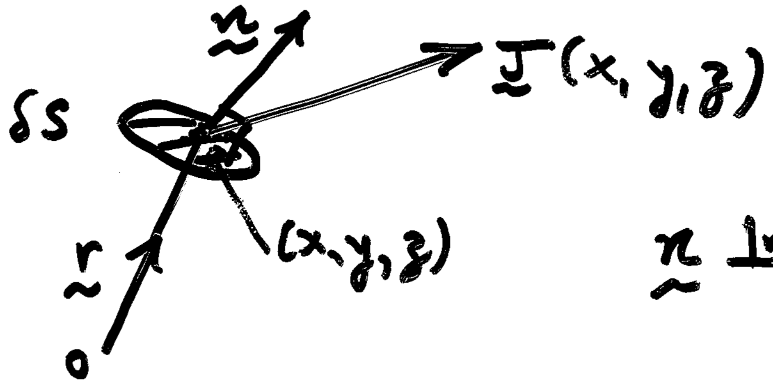
$$= 2\underset{\sim}{\omega}$$

Flux vector field

When have some substance flowing through space, introduce flux vector field

$\underset{\sim}{J}(x,y,z)$ [or $\underset{\sim}{J}(x,y,z,t)$]

Meaning:



$\underset{\sim}{n}$ $\perp$r to δS

$\underset{\sim}{J} \cdot \underset{\sim}{n}\, \delta S$ = amount of substance flowing across δS per unit time.
= flux across δS

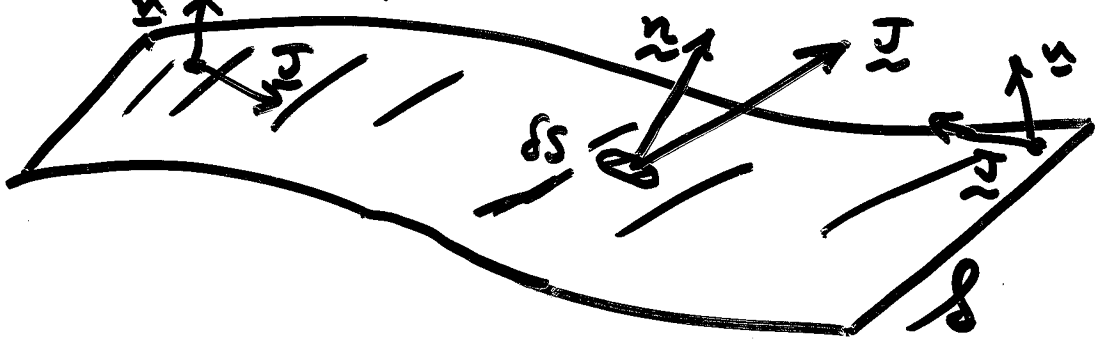
EXS: 1) Moving fluid, velocity field $\underset{\sim}{v}$, density ρ

$\Rightarrow$ mass flux vector $\underset{\sim}{J} = \rho \underset{\sim}{v}$

2) Uniform solid, conducting heat.
Temperature field T
Thermal conductivity κ

$\Rightarrow$ heat flux vector $\underset{\sim}{J} = -\kappa \underset{\sim}{\nabla} T$

When we add the fluxes through all the infinitesimal elements of area making up a surface S, we get the flux over the surface (= amount of substance flowing across the surface per unit time).



We write this total flux as

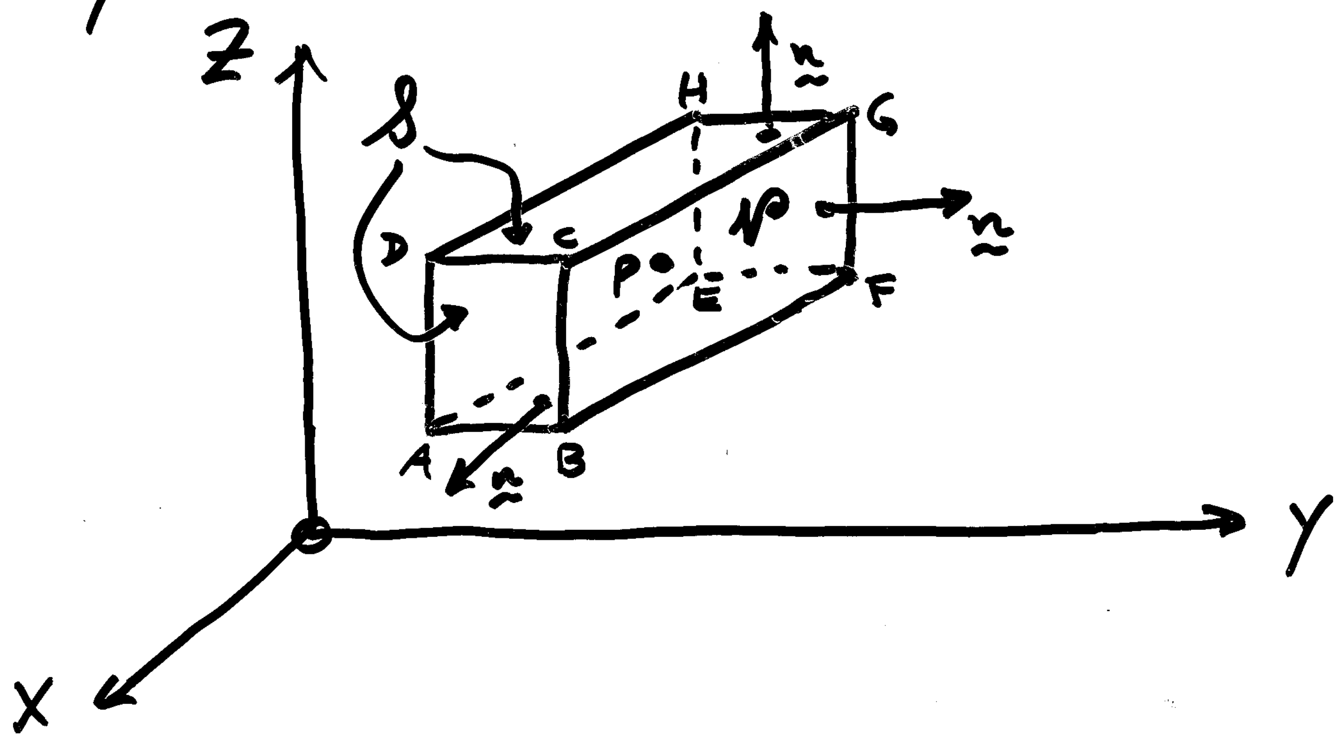
$$\iint_S \underset{\sim}{J}(x,y,z) \cdot \underset{\sim}{n}(x,y,z)\, dS$$

OR simply

$$\boxed{\iint_S \underset{\sim}{J} \cdot \underset{\sim}{n}\, dS \quad \text{Flux Integral}}$$

[Write $\oiint_S \underset{\sim}{J} \cdot \underset{\sim}{n}\, dS$ if S is closed.]

Consider what happens when $\mathcal{S}$ is surface of small, rectangular box, sides parallel to coordinate axes:-



Suppose $P:(x,y,z)$ is at centre of box, and lengths of sides are $|BF| = 2\Delta x$, $|AB| = 2\Delta y$, $|BC| = 2\Delta z$

$\Rightarrow$ The 3-D region $\mathcal{V}$ enclosed by $\mathcal{S}$ has volume $\delta V = 8\Delta x\,\Delta y\,\Delta z$

Estimate flux out of $\mathcal{V}$ over face ABCD:

$$\underset{\sim}{J} \approx \underset{\sim}{J}(x+\Delta x, y, z) \approx \underset{\sim}{J}(x,y,z) + \Delta x\,\frac{\partial \underset{\sim}{J}(x,y,z)}{\partial x}$$

$\underset{\sim}{n} = \underset{\sim}{i}$

$\delta S = 4\Delta y\,\Delta z$

So $\quad \underset{\sim}{J}\cdot\underset{\sim}{n}\,\delta S \approx \left[\underset{\sim}{J}(x,y,z)\cdot\underset{\sim}{i} + \Delta x\,\frac{\partial \underset{\sim}{J}(x,y,z)}{\partial x}\cdot\underset{\sim}{i}\right]4\Delta y\Delta z$

$$= J_1(x,y,z)\,4\Delta y\,\Delta z + \frac{\partial J_1(x,y,z)}{\partial x}\,4\Delta x\,\Delta y\,\Delta z$$

Similarly, flux out of V over face EFGH:

$$\underset{\sim}{J} \approx \underset{\sim}{J}(x-\Delta x, y, z) \approx \underset{\sim}{J}(x,y,z) - \Delta x\,\frac{\partial \underset{\sim}{J}(x,y,z)}{\partial x}$$

$\underset{\sim}{n} = -\underset{\sim}{i} \qquad\qquad \delta S = 4\Delta y\,\Delta z$

$$\Rightarrow \underset{\sim}{J}\cdot\underset{\sim}{n}\,\delta S \approx -J_1(x,y,z)\,4\Delta y\,\Delta z + \frac{\partial J_1(x,y,z)}{\partial x}\,4\Delta x\,\Delta y\,\Delta z$$

So: Flux out over ABCD plus flux out over EFGH $\approx \dfrac{\partial J_1(x,y,z)}{\partial x}\,8\Delta x\,\Delta y\,\Delta z = \dfrac{\partial J_1(x,y,z)}{\partial x}\,\delta V$

Similarly: Flux out over BCGF plus flux out over ADHE $\approx \dfrac{\partial J_2(x,y,z)}{\partial y}\,\delta V$

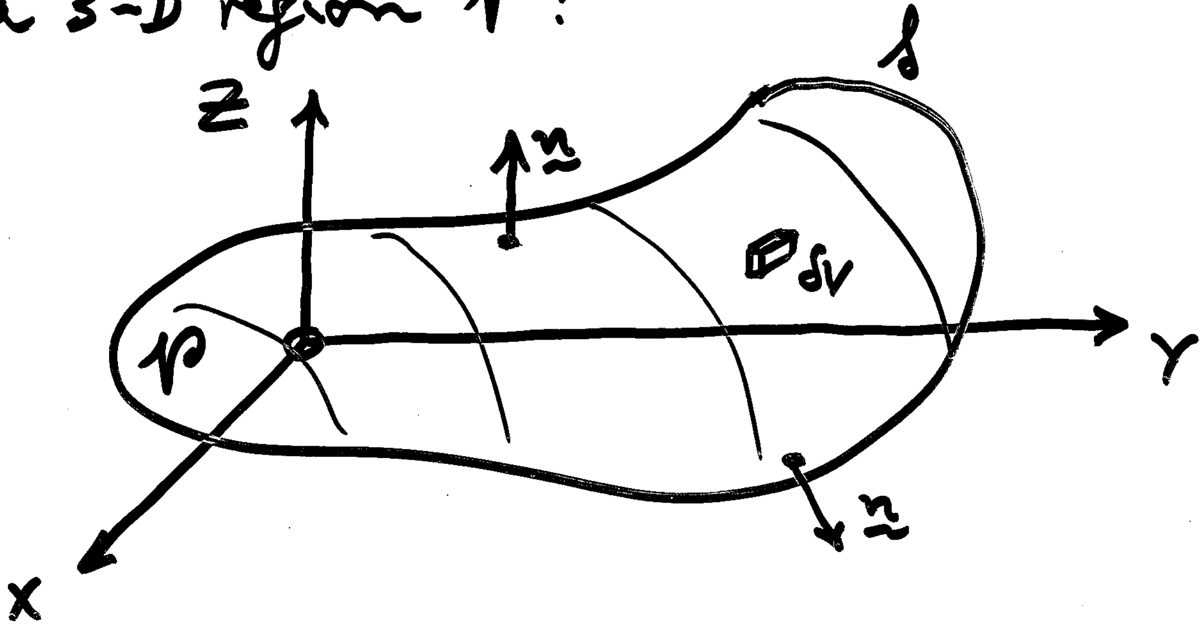
and: Flux out over ABFE plus flux out over DCGH $\approx \dfrac{\partial J_3(x,y,z)}{\partial z}\,\delta V$

and so total flux out of $\mathcal{V}$ over $\mathcal{S}$

$$\approx \left[\frac{\partial J_1(x,y,z)}{\partial x} + \frac{\partial J_2(x,y,z)}{\partial y} + \frac{\partial J_3(x,y,z)}{\partial z}\right]\delta V$$

$$= (\underset{\sim}{\nabla} \cdot \underset{\sim}{J})\,\delta V$$

Now consider a general surface $\mathcal{S}$ enclosing a 3-D region $\mathcal{V}$:



Imagine dicing $\mathcal{V}$ into many tiny rectangular boxes, keeping each in its place.

See sum of outfluxes over all boxes $\approx$ total outflux over $\mathcal{S}$,

because <u>fluxes over internal faces of boxes cancel in pairs</u>.

Now $$\sum_{\text{boxes}} \underset{\sim}{J}\cdot\underset{\sim}{n}\,\delta S \approx \sum_{\text{boxes}} (\underset{\sim}{\nabla}\cdot\underset{\sim}{J})\,\delta V$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\oiint_{\mathcal{S}} \underset{\sim}{J}\cdot\underset{\sim}{n}\,dS \qquad\qquad \iiint_{\mathcal{V}} (\underset{\sim}{\nabla}\cdot\underset{\sim}{J})\,dV$$

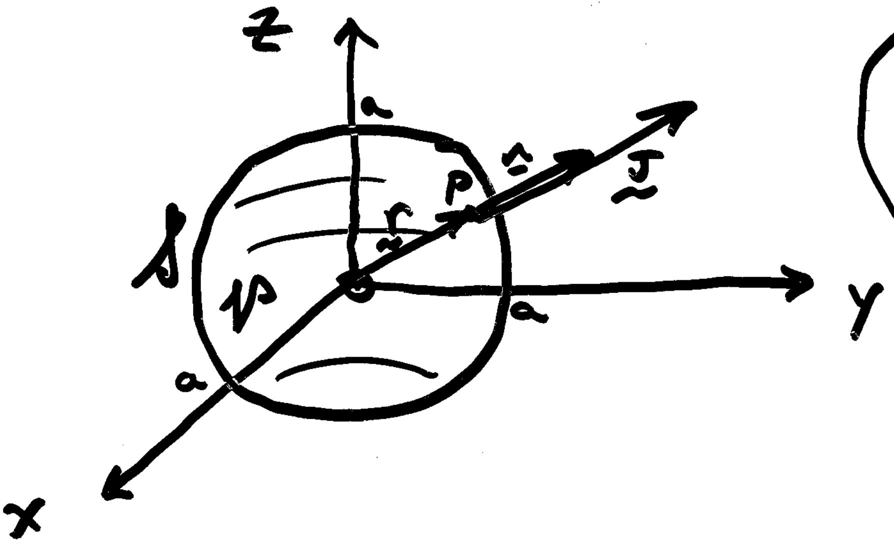
Going to limit of infinitesimal boxes we get:

$$\boxed{\oiint_{\mathcal{S}} \underset{\sim}{J}\cdot\underset{\sim}{n}\,dS = \iiint_{\mathcal{V}} (\underset{\sim}{\nabla}\cdot\underset{\sim}{J})\,dV \quad \left(= \iiint_{\mathcal{V}} \operatorname{div}\underset{\sim}{J}\;dV\right)} \qquad (24.1)$$

Gauss' Divergence Theorem

<u>Ex:</u> $\underset{\sim}{J}(x,y,z) = \alpha(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) = \alpha\underset{\sim}{r}$ $(\alpha > 0)$

$(\Rightarrow \underset{\sim}{\nabla}\cdot\underset{\sim}{J} = \alpha + \alpha + \alpha = 3\alpha)$



Take $\mathcal{S}$ = sphere, radius a centre O.

At any P on $\mathcal{S}$, see $\underset{\sim}{r}, \underset{\sim}{n}, \underset{\sim}{J}$ all radially outwards.

Then, on $\mathcal{S}$, $\underset{\sim}{J}\cdot\underset{\sim}{n} = \alpha\underset{\sim}{r}\cdot\hat{\underset{\sim}{r}} = \alpha|\underset{\sim}{r}| = \alpha a$

Then $\oiint_{\mathcal{S}} \underset{\sim}{J}\cdot\underset{\sim}{n}\, dS = \alpha a \oiint_{\mathcal{S}} dS = \alpha a(4\pi a^2) = 4\pi\alpha a^3$

and $\iiint_{\mathcal{V}} (\underset{\sim}{\nabla}\cdot\underset{\sim}{J})\, dV = 3\alpha \iiint_{\mathcal{V}} dV = 3\alpha\left(\frac{4}{3}\pi a^3\right) = 4\pi\alpha a^3$

It works!

Summary:

1) Understand definitions and meanings of grad φ, div $\underset{\sim}{a}$ and curl $\underset{\sim}{a}$.

2) Understand notion of a flux vector field.

3) Understand notion of a flux integral

4) Know Gauss' Divergence Theorem (not proof).

~~K p. 481, §9.7, 9.8~~

K §10.6, 10.7, 10.8