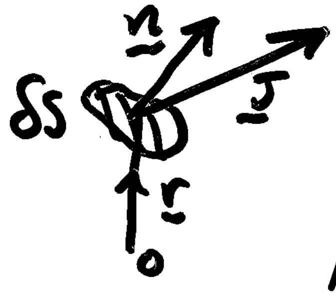
MATH2100 Lec. 25 (= MATH2011 Lec. 7)

A conservation law:

Consider some substance flowing through space, with density $\rho(x,y,z,t)$ and flux vector field $\underset{\sim}{J}(x,y,z,t)$:-

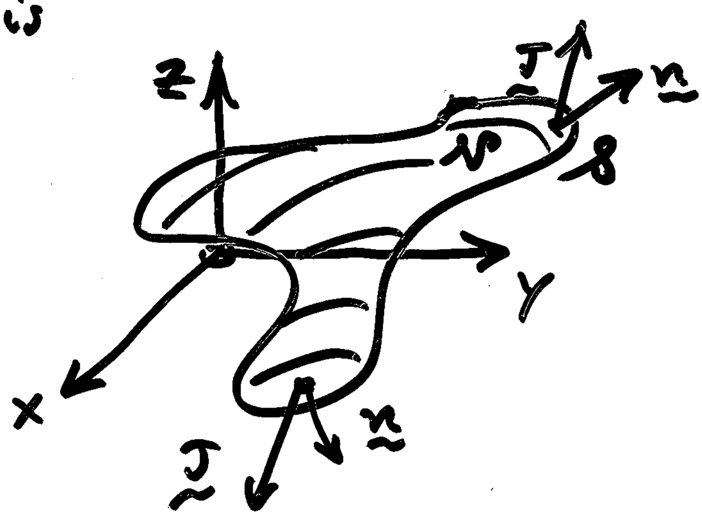
Amount of substance in δV at (x,y,z) at time t $= \rho(x,y,z,t)\,\delta V$



Amount of substance flowing per unit time over δS at (x,y,z)
$= \underset{\sim}{J}(x,y,z,t)\cdot\underset{\sim}{n}\,\delta S$

If substance is <u>conserved</u> (not being created or destroyed), then for <u>every</u> closed surface S, and corresponding enclosed 3-D region V, we must have

$$\underbrace{\frac{d}{dt}\underbrace{\iiint_V \rho \, dV}_{\substack{\text{amount in } V \\ \text{at time } t}}}_{\substack{\text{rate at which} \\ \text{amount in } V \text{ is} \\ \text{increasing}}} = \underbrace{-\oint\!\!\oint_S \underline{J}\cdot\underline{n} \, dS}_{\substack{\text{flux } \underline{\text{into}} \ V \text{ over } S \\ \text{per unit time}}}$$



$$\Rightarrow \iiint_V \frac{\partial \rho}{\partial t} \, dV = -\iiint_V (\underline{\nabla}\cdot\underline{J}) \, dV$$

using Gauss' Theorem

$$\Rightarrow \iiint_{V} \left(\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} \right) dV = 0$$

Since S and corresponding V are arbitrary, follows that at every point (x, y, z) we have

$$\boxed{\frac{\partial \rho(x,y,z,t)}{\partial t} + \underset{\sim}{\nabla} \cdot \underline{J}(x,y,z,t) = 0} \quad (25.1)$$

Conservation of 'substance'

EX: 1) Moving fluid, velocity field $\underset{\sim}{v}(x,y,z,t)$

If fluid is homogeneous & incompressible, mass density $\rho(x,y,z,t) = \rho_0$ (const.)

Since mass flux vector field is $\rho_0 \underset{\sim}{v}$, we have

$$\underbrace{\cancel{\frac{\partial \rho_0}{\partial t}}}_{0} + \underset{\sim}{\nabla} \cdot (\rho_0 \underset{\sim}{v}) = 0 \Rightarrow \rho_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$$

$$\Rightarrow \boxed{\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0}$$

Conservation of mass of incompressible fluid.

EX: 2)

Important example for rest of course:-

Conduction of heat in a uniform solid.

$$u(x,y,z,t) = \text{temperature at } (x,y,z) \text{ at time } t$$

ρ = mass/unit volume of solid

σ = specific heat of solid (= amount of heat energy to raise unit mass through one unit of temperature)

$\Rightarrow$ $\rho\sigma\, u(x,y,z,t)$ = heat energy density

Heat energy flux vector

$$\underset{\sim}{J}(x,y,z,t) = -\underbrace{\kappa}_{\text{thermal conductivity}} \underset{\sim}{\nabla} u(x,y,z,t)$$

[Heat flows from hotter to cooler regions, "down the temperature gradient"]

Conservation of heat energy:

$$\frac{\partial}{\partial t}\left[\rho\sigma u(x,y,z,t)\right] + \underset{\sim}{\nabla}\cdot\left[-\kappa\underset{\sim}{\nabla}u(x,y,z,t)\right] = 0$$

$$\Rightarrow \quad \frac{\partial u(x,y,z,t)}{\partial t} - \underbrace{\frac{\kappa}{\rho\sigma}}_{k=c^2}\underset{\sim}{\nabla}\cdot\left[\underset{\sim}{\nabla}u(x,y,z,t)\right] = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial u(x,y,z,t)}{\partial t} = c^2\,\underset{\sim}{\nabla}\cdot\left[\underset{\sim}{\nabla}u(x,y,z,t)\right]} \qquad (25.2)$$

The heat (conduction) equation.

$\boxed{c^2 : \text{thermal diffusivity}}$

Consider $\underset{\sim}{\nabla}\cdot\left[\underset{\sim}{\nabla}u\right] = \underset{\sim}{\nabla}\cdot\left(\frac{\partial u}{\partial x}\underset{\sim}{i} + \frac{\partial u}{\partial y}\underset{\sim}{j} + \frac{\partial u}{\partial z}\underset{\sim}{k}\right)$

$$= \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial u}{\partial x}\underset{\sim}{i} + \frac{\partial u}{\partial y}\underset{\sim}{j} + \frac{\partial u}{\partial z}\underset{\sim}{k}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Write this as $\nabla^2 u \quad (= \text{del}^2\, u)$

$$\nabla^2 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right) \cdot \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\underset{\sim}{\nabla}$ vector differential operator

$\nabla^2 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla}$ scalar differential operator

∇^2 called "the Laplacian"

So the heat equation (25.2) says

$$\boxed{\begin{aligned} \frac{\partial u(x,y,z,t)}{\partial t} &= c^2\left[\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right] \\ &= c^2 \nabla^2 u(x,y,z,t) \qquad (25.3) \end{aligned}}$$

A partial differential equation (PDE)

4 independent variables x, y, z, t

1 dependent variable u

1st-order in t 2nd order in x, y and z

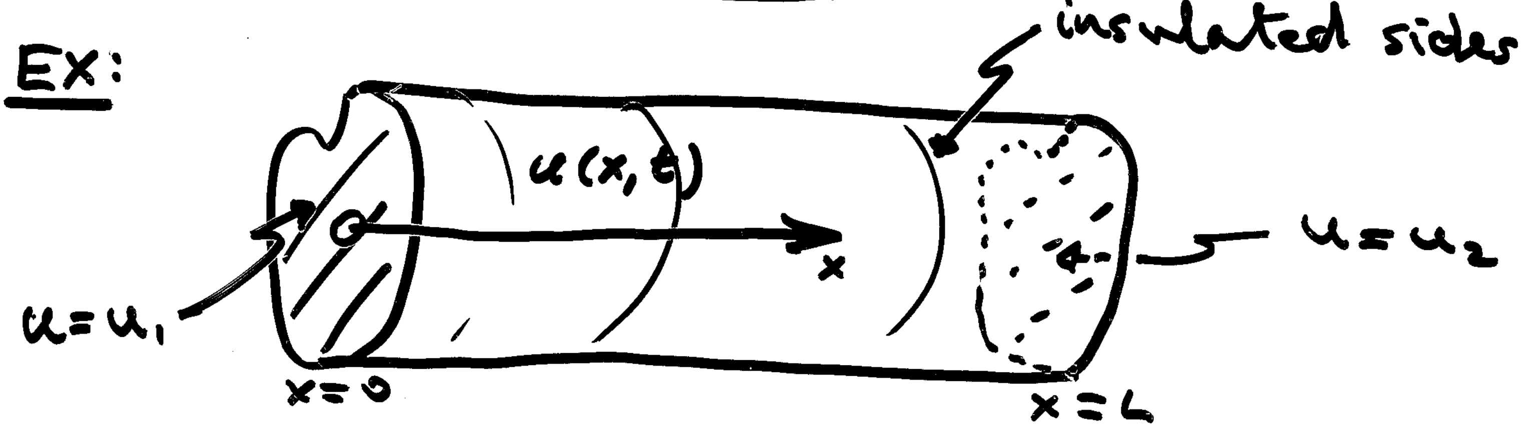
Sometimes call (25.3) the <u>3-D heat equation</u>

If no y or z dependence in problem, get

$$\boxed{\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (25.4)}$$

<u>1-D heat equation.</u>

<u>EX:</u>



iron bar, initially at constant temperature u_0. Then one face held at temp. $u=u_1$, other at $u=u_2$.

$u(x,t)$ satisfies (25.4) for $t>0$ and $0<x<L$.

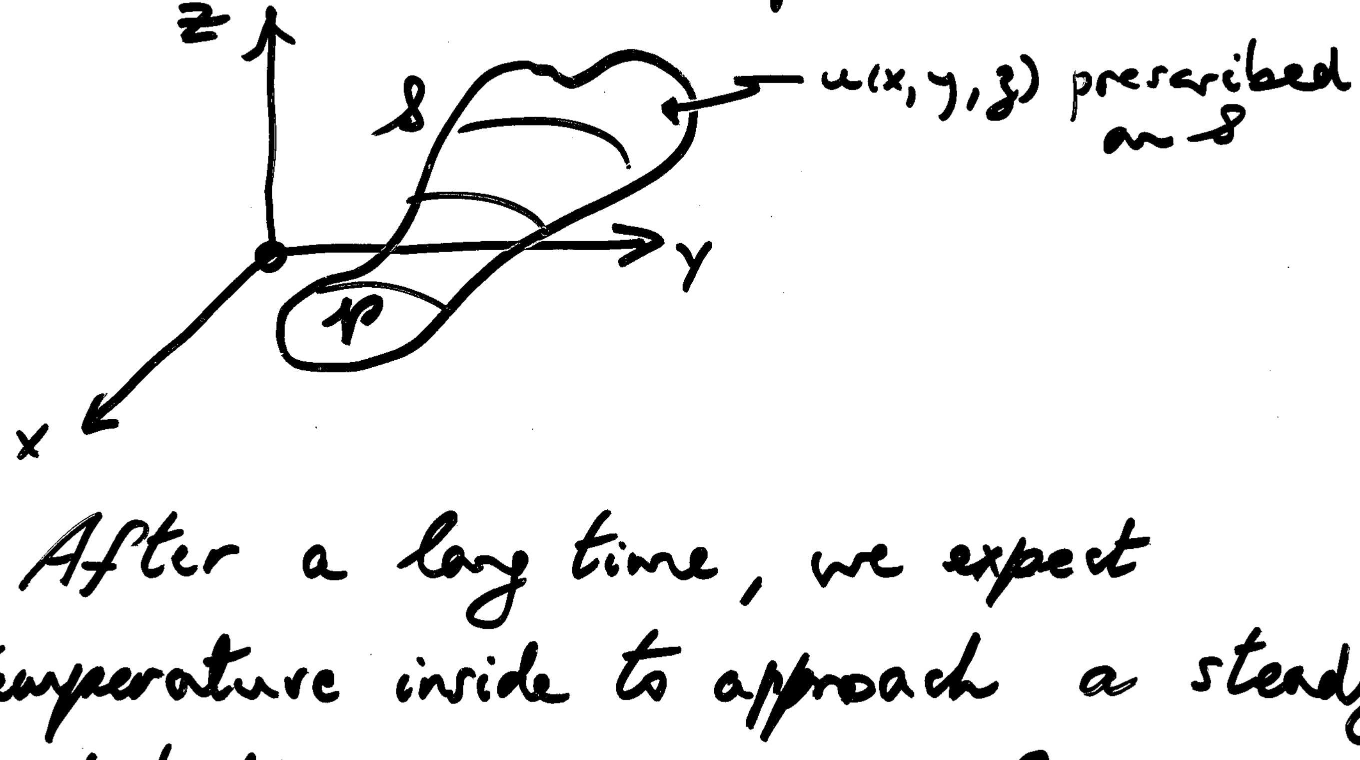
Typical boundary/initial value problem:-

Find solution $u(x,t)$ with IC $u(x,0)=u_0$

BCs $u(0,t)=u_1$, $u(L,t)=u_2$.

The heat equation governs the way temperature varies as heat flows from hotter to cooler regions, in a way consistent with conservation of heat energy.

Consider a 3-D region, with a prescribed, constant (in time) temperature distribution on the surface:—



After a long time, we expect temperature inside to approach a steady distribution $u(x,y,z)$ satisfying (from (25.3))

$$\cancel{\frac{\partial u(x,y,z)}{\partial t}}_{0} = c^2 \nabla^2 u(x,y,z)$$

$$\Rightarrow \boxed{\begin{gathered}\nabla^2 u(x,y,z) = 0 \\ \frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0\end{gathered}} \quad (25.5)$$

Laplace's Equation

- describes equilibrium (steady-state) temperature distributions.

If no z-dependence in problem,

$$\boxed{\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \quad (25.6)}$$

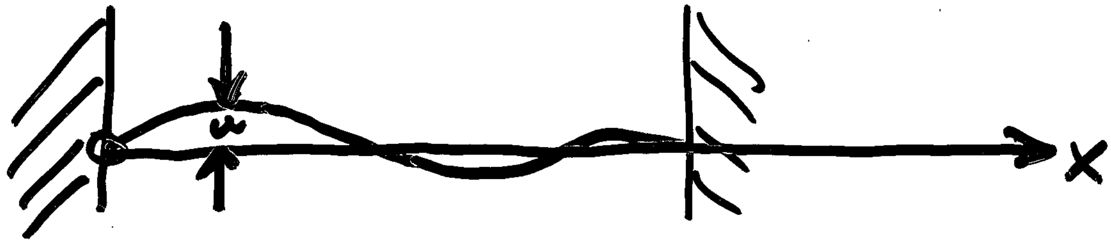
2-D Laplace Equation

PDE with 2 indep. variables x, y and one dependent variable u.

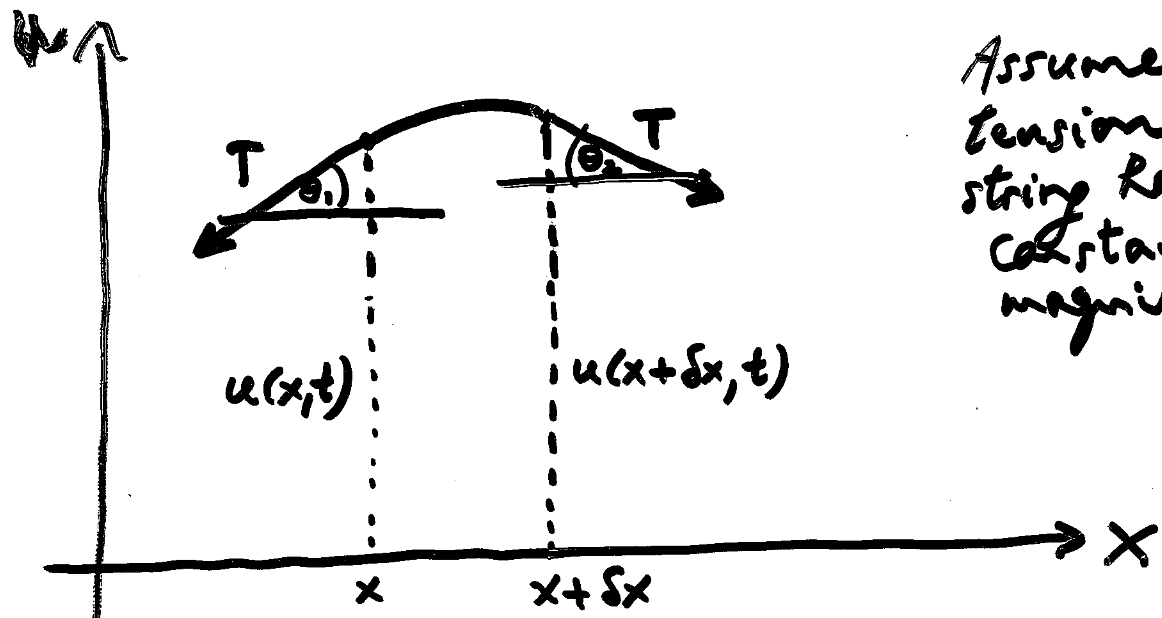
One more PDE with 2 indep. variables and one dep. variable for us to consider:

Consider small displacements of stretched string, whose equilibrium position is along x-axis:-

25.10



Consider small segment:-



Assume tension T in string has constant magnitude

Force on segment in $+u$-direction

$$\approx -T\sin\theta_1 - T\sin\theta_2$$

$$\approx -T\tan\theta_1 - T\tan\theta_2 \qquad (|\theta_1|, |\theta_2| << 1)$$

$$= -T\frac{\partial u(x,t)}{\partial x} + T\frac{\partial u(x+\delta x, t)}{\partial x}$$

$$\approx -T\frac{\partial u(x,t)}{\partial x} + T\left[\frac{\partial u(x,t)}{\partial x} + \delta x\,\frac{\partial^2 u(x,t)}{\partial x^2}\right]$$

$$\approx T\frac{\partial^2 u(x,t)}{\partial x^2}\,\delta x$$

If mass/unit length of string is ρ $= \text{const}$, then mass × accel. of segment in u-direction is

$$(\rho \delta x) \frac{\partial^2 u(x,t)}{\partial t^2}$$

Then, by Newton's 2nd Law,

$$T \frac{\partial^2 u(x,t)}{\partial x^2} \delta x \approx \rho \delta x \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad (25.7) \qquad c^2 = \frac{T}{\rho}}$$

1-D wave equation

[The 3-D wave equation is

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2 \nabla^2 u(x,y,z,t)$$]

We will consider the 3 PDEs in one dependent, and two independent variables:-

The 1-D heat equation:

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

Laplace's Equation in 2-D:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

The 1-D wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

Summary:

1) Understand idea of a conservation law and know (25.1)

2) Know definition of Laplacian operator ∇^2

3) Know forms of heat eqn, wave eqn and Laplace's eqn in 1, 2, 3 dimensions

~~K pp 511, 512.~~ ~~§11.1, 11.2~~
~~p 451~~

K pp. 407, 408, 811
458-5
§12.1, 12.2