MATH2100 Lec. 26 (= MATH2011 Lec. 8)

The 3 PDEs

$$\frac{\partial u(x,t)}{\partial t} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad (26.1)$$

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad (26.2)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad (26.3)$$

are all <u>linear</u> and <u>homogeneous</u> $\Rightarrow$ the <u>superposition principle</u> applies in each case:

If $u = u_1$ and $u = u_2$ satisfy (26.1), so does

$$u = \alpha u_1 + \beta u_2$$

α, β arbitrary constants.

Similarly for (26.2) and also for (26.3).

PDEs have a much richer set of solutions than ODEs:

Easy to check that, for example,

$$u_1 = e^{-\alpha^2 t} \sin\left(\frac{\alpha x}{c}\right), \quad u_2 = x^3 + 6c^2 x t$$

are each solutions of (26.1);

$$u_1 = \cos(\alpha x)\sinh(\alpha y), \quad u_2 = e^{-\alpha x}\sin(\alpha y),$$

$$u_3 = \alpha(x^3 - 3xy^2)$$

are each solutions of (26.2); and

$$u_1 = x^2 + c^2 t^2, \quad u_2 = e^{\alpha(x - ct)}$$

$$u_3 = \sin(x)\cos(ct) \qquad (26.4)$$

are each solutions of (26.3). It is not so easy to find <u>general</u> solutions. But we <u>can</u> do it for the 1-D wave equation, (26.3).

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

We change the independent variables:

Let $v = x + ct \qquad z = x - ct$

$\left(\leftrightarrow \quad x = \frac{1}{2}(v+z), \quad t = \frac{1}{2c}(v-z) \right)$

Then

$$\frac{\partial}{\partial x} = \frac{\partial v}{\partial x}\frac{\partial}{\partial v} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial v} + \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial t} = \frac{\partial v}{\partial t}\frac{\partial}{\partial v} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = c\frac{\partial}{\partial v} - c\frac{\partial}{\partial z}$$

Then

$$\frac{\partial u^2}{\partial t^2} = \left(c\frac{\partial}{\partial v} - c\frac{\partial}{\partial z}\right)\left(c\frac{\partial}{\partial v} - c\frac{\partial}{\partial z}\right)u$$

$$= c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial v} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}\right)$$

So $$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}\right) - c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial z \partial v} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$$\Rightarrow \quad -4c^2 \frac{\partial^2 u}{\partial z \partial v} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial^2 u}{\partial z \partial v} = 0} \qquad (26.5)$$

(26.5) is <u>equivalent</u> to (26.3) – but can see how to solve (26.5)!

$$\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial v}\right] = 0$$

$$\Rightarrow \frac{\partial u}{\partial v} = A(v) \qquad A \text{ arbitrary fn.}$$

$$u = \underbrace{\int A(v)\,dv}_{\text{"}F(v)\text{, arbitrary}} + G(z) \qquad G \text{ arbitrary fn.}$$

So $$u = F(v) + G(z)$$

$$\Rightarrow u(x,t) = F(x+ct) + G(x-ct) \qquad F, G \text{ arbitrary fns. (twice diff'ble)}$$

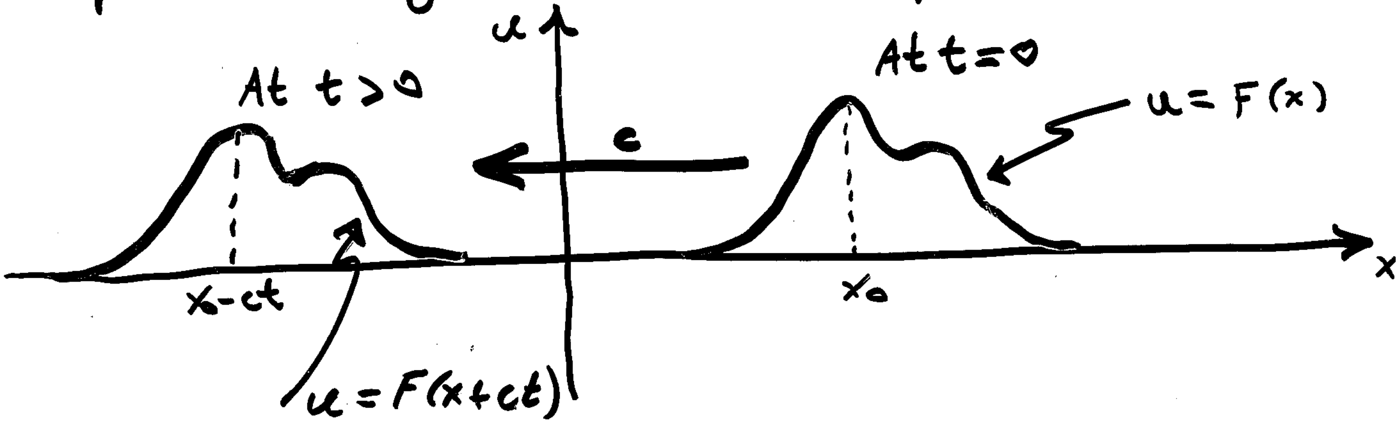
Check each of (26.4) can be written this way!

General solution of 1-D wave equation:

$$u(x,t) = F(x+ct) + G(x-ct) \qquad (26.6)$$

F, G arbitrary fns. of a single variable

__Meaning__: $F(x+ct)$ represents a wave of constant shape travelling to __left__ at speed c:



Similarly $G(x-ct)$ represents a wave of constant shape travelling to __right__ at speed c.

We call c the __wave speed__. $\left[\text{Recall } c = \sqrt{\frac{T}{\rho}}\right]$

To fix a __particular__ solution of the wave equation, we have to give enough info. to fix the two functions F and G.

Initial Value Problem (IVP) for infinite string.

Suppose string is very long, and we are looking near middle — ends so far away we can ignore end effects (= boundary conditions).
In effect, we consider

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 , \qquad \begin{matrix} -\infty < x < \infty \\ t > 0 \end{matrix}$$

Aside on notation: Let's agree to write

$$\frac{\partial u(x,t)}{\partial t} = u_t(x,t) , \quad \frac{\partial u(x,t)}{\partial x} = u_x(x,t),$$

$$\frac{\partial^2 u(x,t)}{\partial t \partial x} = u_{tx}(x,t), \quad \frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x,t)$$

and so on.

So our PDE is $\quad u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0$

Now suppose we have ICs:

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \text{given fns.,} \quad -\infty < x < \infty$$

f: initial shape $\qquad g$: initial velocity profile

We know solution of PDE + ICs must be of form

$$u(x,t) = F(x+ct) + G(x-ct) \qquad (26.7)$$

for some F and G. Can we determine F and G from given f and g?

Well, (26.7) $\Rightarrow$ $u_t(x,t) = \dfrac{\partial F(x+ct)}{\partial t} + \dfrac{\partial G(x-ct)}{\partial t}$

$$= \frac{\partial (x+ct)}{\partial t} F'(x+ct) + \frac{\partial(x-ct)}{\partial t} G'(x-ct)$$

i.e. $$u_t(x,t) = cF'(x+ct) - cG'(x-ct) \qquad (26.8)$$

Now, using ICs at $t=0$:

(26.7) $\Rightarrow$ $$f(x) = u(x,0) = F(x) + G(x) \qquad (26.9)$$

(26.8) $\Rightarrow$ $$g(x) = u_t(x,0) = cF'(x) - cG'(x) \qquad (26.10)$$

Integrating (26.10) w.r.t. x we get:–

$$cF(x) - cG(x) = \int g(x)\,dx$$

or better,

$$F(x) - G(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + A \qquad (26.11)$$

Now can use (26.9) and (26.11) to fix F and G:

Adding: $$2F(x) = f(x) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + A$$

Subtracting: $$2G(x) = f(x) - \frac{1}{c}\int_{x_0}^{x} g(s)\,ds - A$$

$$\Rightarrow \quad F(x+ct) + G(x-ct) = \tfrac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds + \tfrac{1}{2}A$$

$$+ \tfrac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds - \tfrac{1}{2}A$$

Now $\int_a^b \cdots - \int_a^c \cdots = \int_c^b \cdots$, so

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$\Rightarrow \quad \boxed{u(x,t) = \tfrac{1}{2}f(x+ct) + \tfrac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds} \quad (26.12)$$

D'Alembert's solution for IVP of 1-D wave equation on whole x-axis.

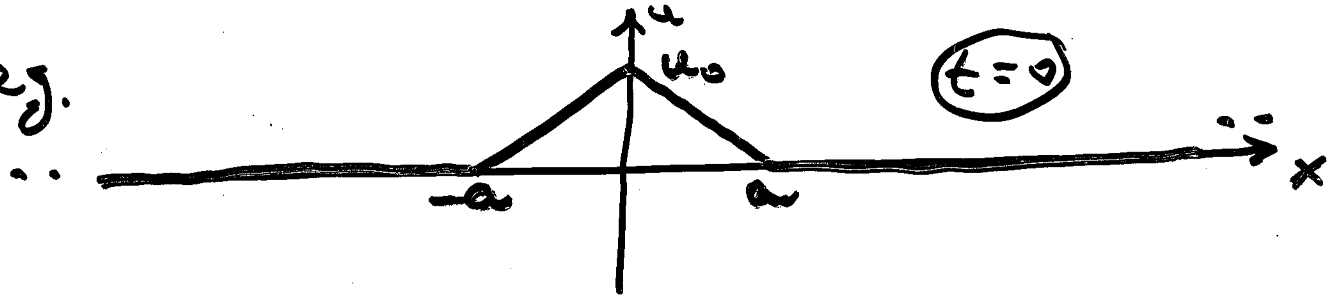
Ex: 1 ICs $u(x,0) = f(x)$, $u_t(x,0) = 0 = g(x)$

(string released from rest with initial shape given by $f(x)$)

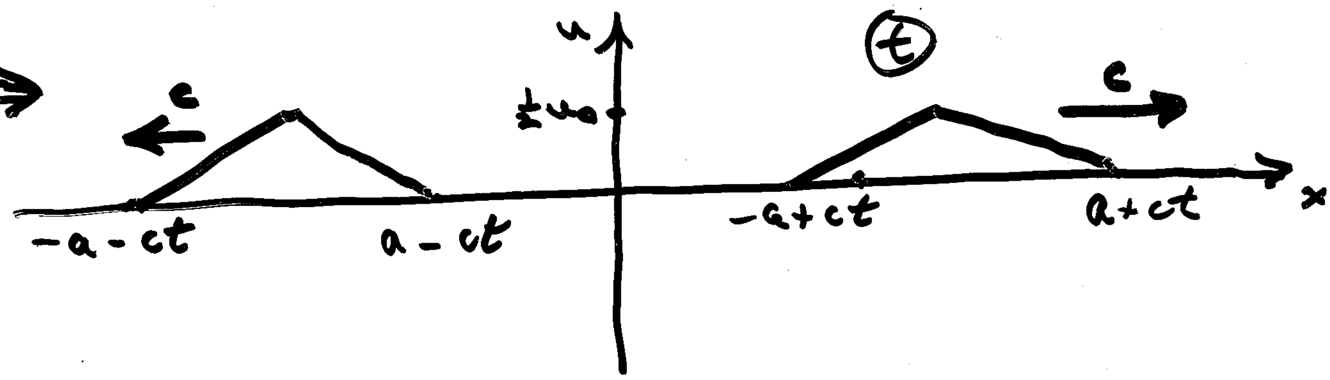
$$\Rightarrow \quad u(x,t) = \tfrac{1}{2} f(x+ct) + \tfrac{1}{2} f(x-ct)$$

Get pulses to left and right, each at speed c, each with shape $\tfrac{1}{2}f$:

e.g.



$\Rightarrow$

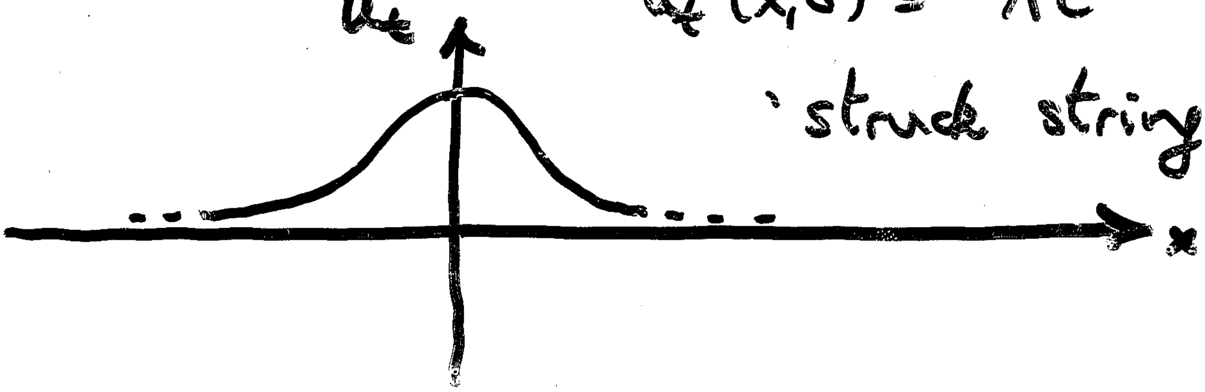


'Plucked string'

EX: 2 ICs: $u(x,0) = 0 = f(x)$

$u_t(x,0) = Ae^{-x^2} = g(x)$

'struck string'



D'Alembert's solution:

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} Ae^{-s^2}\,ds$$

— can't evaluate

$$= \frac{A\sqrt{\pi}}{4c}\left\{\frac{2}{\sqrt{\pi}}\int_0^{x+ct} e^{-s^2}\,ds - \frac{2}{\sqrt{\pi}}\int_0^{x-ct} e^{-s^2}\,ds\right\}$$

Introduce new function:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^{z} e^{-s^2}\,ds$$

— the <u>error function</u>

Then our solution is

$$u(x,t) = \underbrace{\frac{A\sqrt{\pi}}{4c}}_{=B,\ \text{say}} \left\{ \operatorname{erf}(x+ct) - \operatorname{erf}(x-ct) \right\}$$

How does it look? How does erf look?

Some properties of $\operatorname{erf}(z)$:

1) $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ <u>odd function</u>

Pf: $\operatorname{erf}(-z) = \frac{2}{\sqrt{\pi}} \int_0^{-z} e^{-s^2}\, ds$

(Put $v = -s$
$dv = -ds$
$s = -z \Leftrightarrow v = z$
$s = 0 \Leftrightarrow v = 0$)

$$= \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-v^2} (-dv)$$

$$= -\operatorname{erf}(z).$$

2) $\frac{d}{dz} \operatorname{erf}(z)\ \left(\equiv \operatorname{erf}'(z)\right) = \frac{2}{\sqrt{\pi}} e^{-z^2} > 0$

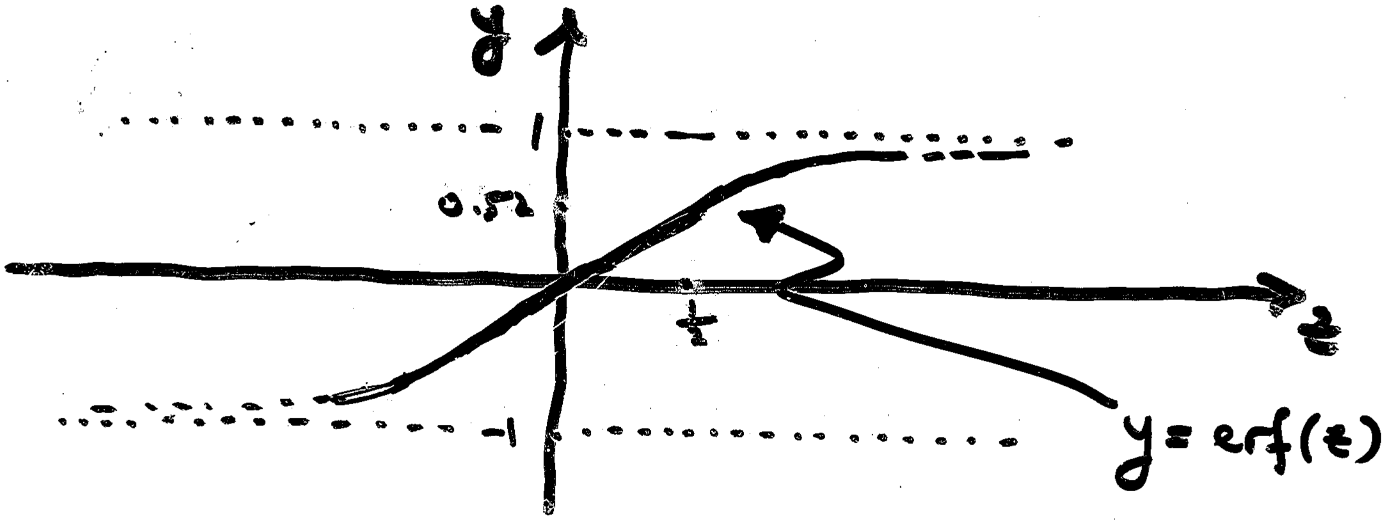
$\Rightarrow \operatorname{erf}(z)$ is <u>monotonically increasing</u>

3) $\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = 1$

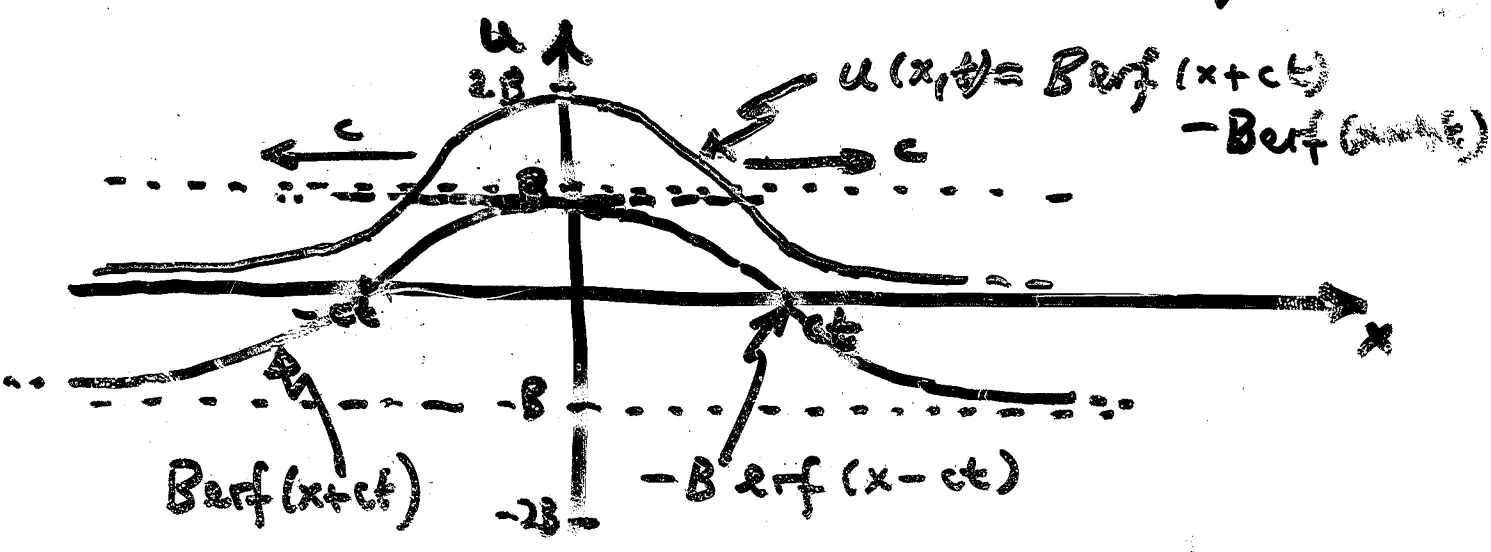
(Then 1) $\Rightarrow \operatorname{erf}(-\infty) = -1$)

4) $\operatorname{erf}(0) = 0$

5) $\operatorname{erf}(\frac{1}{2}) \approx 0.52$, $\operatorname{erf}(1) \approx 0.84$,
$\operatorname{erf}(2) \approx 0.995$, $\operatorname{erf}(3) \approx 0.99998$



So, our solution to 1-D Wave Equation:



Summary:

1) PDEs have wide variety of solutions

2) Know (26.1, 2, 3). They are linear, homogeneous PDEs – superposition principle applies!

3) Know derivation of general solution of 1-D wave Equation and of D'Alembert's solution of IVP.

4) Understand last two examples.

~~K § 11.4, 11.1, 11.2~~

K § 12.1, 12.2, 12.4