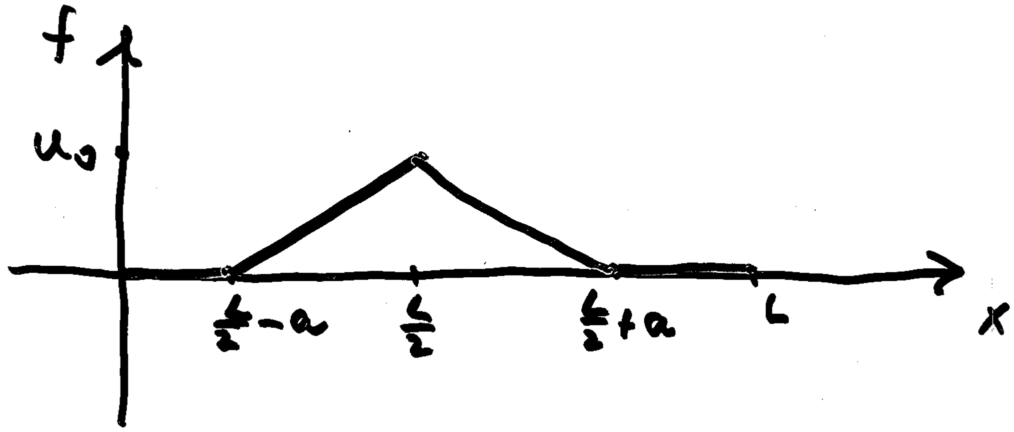
MATH2100 Lec. 28 (= MATH2011 Lec. 10)

EX: $u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < L, \quad t > 0$

BCs $u(0,t) = 0 = u(L,t) \qquad t > 0$

IC
$$u(x,0) = f(x) = \begin{cases} 0, & \frac{L}{2}+a < x < L \\ 0, & 0 < x < \frac{L}{2}-a \\ u_0(x - \frac{L}{2}+a)/a, & \frac{L}{2}-a < x < \frac{L}{2} \\ u_0(\frac{L}{2}+a-x)/a, & \frac{L}{2} < x < \frac{L}{2}+a \end{cases}$$



IC $u_t(x,0) = g(x) = 0, \qquad 0 < x < L$

Our general result is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

where

$$A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx = \frac{2}{L}\int_{\frac{L}{2}-a}^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right) \frac{u_0(x-\frac{L}{2}+a)}{a}\,dx$$

$$+ \frac{2}{L}\int_{\frac{L}{2}}^{\frac{L}{2}+a} \sin\left(\frac{n\pi x}{L}\right) \frac{u_0(\frac{L}{2}+a-x)}{a}\,dx$$

$$= \frac{4u_0 L}{a n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right)\left\{1-\cos\left(\frac{n\pi a}{L}\right)\right\}$$ check!!

and

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

then

$$u(x,t) = \frac{4u_0 L}{a\pi^2}\Bigg[\left[1-\cos\left(\frac{\pi a}{L}\right)\right]\cos\left(\frac{\pi c t}{L}\right)\sin\left(\frac{\pi x}{L}\right)$$

$$-\frac{1}{9}\left[1-\cos\left(\frac{3\pi a}{L}\right)\right]\cos\left(\frac{3\pi c t}{L}\right)\sin\left(\frac{3\pi x}{L}\right)$$

$$+\frac{1}{25}\left[1-\cos\left(\frac{5\pi a}{L}\right)\right]\cos\left(\frac{5\pi c t}{L}\right)\sin\left(\frac{5\pi x}{L}\right)$$

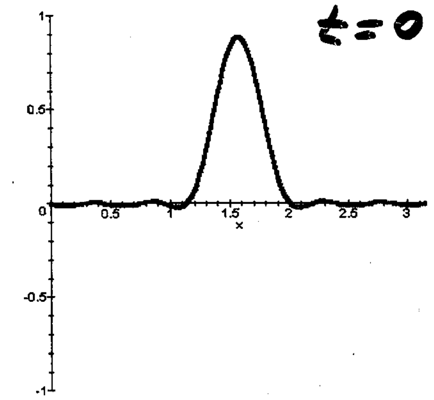
$$\vdots \qquad \Bigg]$$

28.3

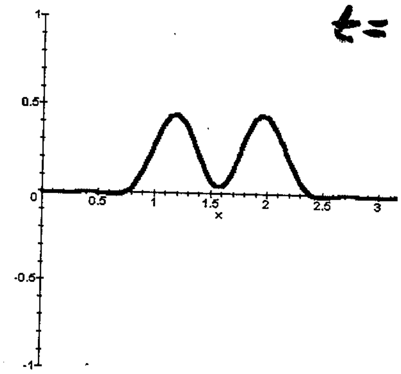
$$u(x,t) = \frac{4u_0 L}{a\pi^2} \sum_{m=0}^{\cancel{\infty}\ 4} \frac{(-1)^m}{(2m+1)^2} \left[1-\cos\frac{(2m+1)\pi a}{L}\right] \cos\frac{(2m+1)\pi ct}{L} \sin\frac{(2m+1)\pi x}{L}$$

$$\left[u_0 = 1,\ L = \pi,\ c = 1,\ a = \frac{L}{8}\right]$$

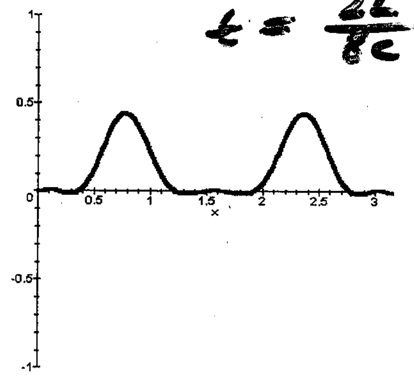
$t = 0$



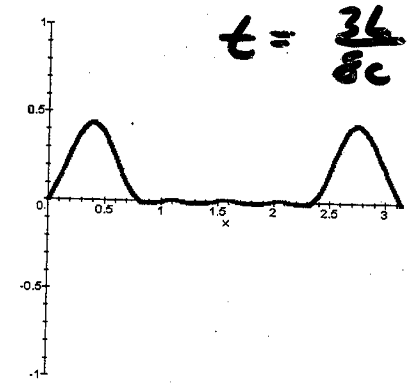
$t = \frac{L}{8c}$



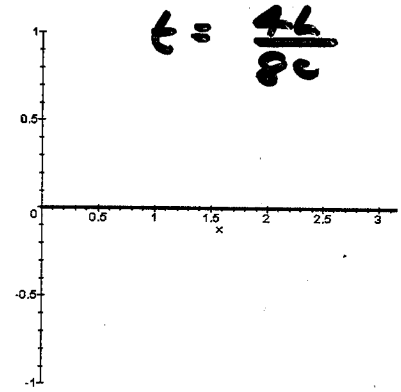
$t = \frac{2L}{8c}$



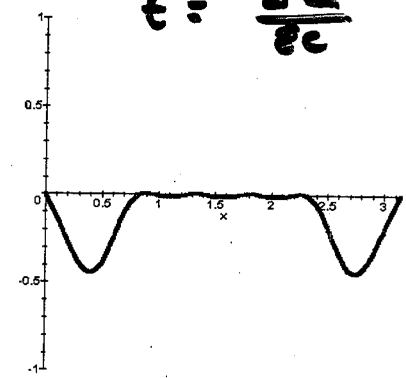
$t = \frac{3L}{8c}$



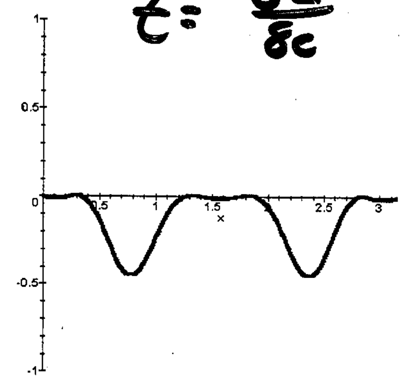
$t = \frac{4L}{8c}$



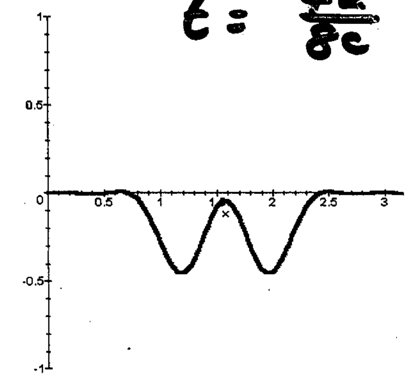
$t = \frac{5L}{8c}$



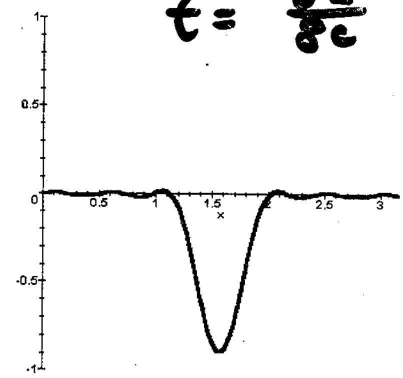
$t = \frac{6L}{8c}$



$t = \frac{7L}{8c}$



$t = \frac{8L}{8c}$



Note: In general

(27.13): $$u(x,t) = \sum_{n=1}^{\infty} \left[A_n\, u_n^{(A)}(x,t) + B_n\, u_n^{(B)}(x,t) \right]$$

where

$$u_n^{(A)}(x,t) = \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left\{ \sin\left[\frac{n\pi}{L}(x+ct)\right] + \sin\left[\frac{n\pi}{L}(x-ct)\right] \right\}$$

$$u_n^{(B)}(x,t) = \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = -\frac{1}{2}\left\{ \cos\left[\frac{n\pi}{L}(x+ct)\right] - \cos\left[\frac{n\pi}{L}(x-ct)\right] \right\}$$

– so $u(x,t)$ is indeed of form $P(x+ct) + Q(x-ct)$

For a problem with $f(x)=0$ (initial displacement) and $g(x) \neq 0$ (initial velocity profile), try

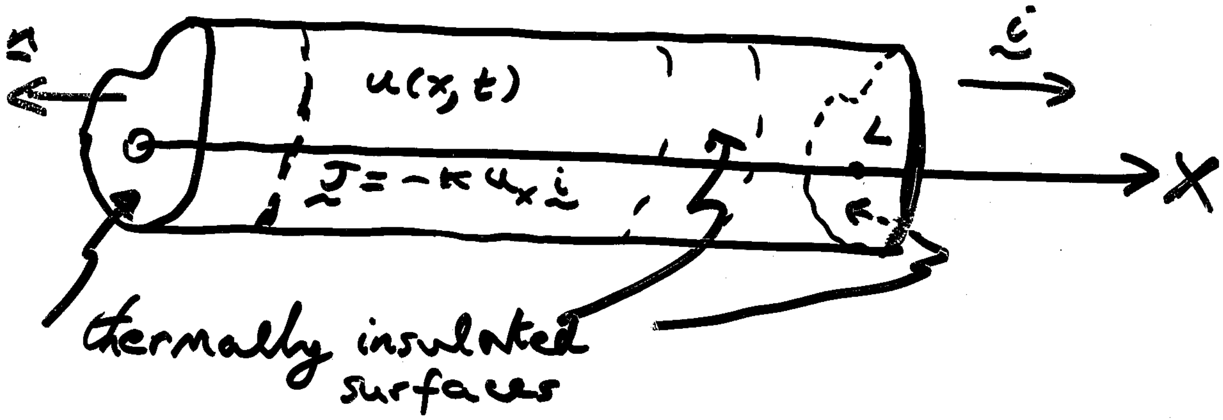
~~K11.3 p.594 #10~~ K12.3 p547 #12

or else $$g(x) = \begin{cases} 0, & 0 < x < \frac{L}{2} - a \\ 0, & \frac{L}{2} + a < x < L \\ g_0, & \frac{L}{2} - a < x < \frac{L}{2} + a \end{cases}$$

(model of struck string: cf. pp 26.9–26.12)

Fourier's Method for the 1-D Heat Equation

Consider iron (say) bar:-



No heat flow through face at $x=0$ $\quad$ $\underset{\sim}{n} = -\underset{\sim}{i}$

$$\Rightarrow \quad -\kappa u_x(0,t)\,\underset{\sim}{i}\cdot\underset{\sim}{i}\,(\text{area of face}) = 0$$

$$\Rightarrow \quad u_x(0,t) = 0$$

Similarly $\quad u_x(L,t) = 0$ $\Big\}$ BCs.

So we have:

PDE $\quad u_t(x,t) = c^2 u_{xx}(x,t) \quad$ (28.1) $\quad \left(\begin{matrix} 0 < x < L \\ t > 0 \end{matrix}\right)$

BCs $\quad u_x(0,t) = 0$ — (28.2a) $\quad u_x(L,t) = 0$ (28.2b) $\quad (t > 0)$

IC $\quad u(x,0) = f(x)$ — (28.3) $\quad (0 < x < L)$

(initial temperature distribution)

Initial- and boundary-value-problem for 1-D Heat Eqn.

We look for solutions of homogeneous equations (28.1), (28.2a, b) in separated form:

$$\boxed{u(x,t) = F(x)\,G(t)} \qquad (28.4)$$

(28.2a) $\Rightarrow \quad 0 = u_x(0,t) = F'(0)\,G(t)$

$\Rightarrow \quad F'(0) = 0 \quad$ (else $G(t) = 0 \Rightarrow u(x,t) = 0$ trivial)

Similarly,

(28.2b) $\Rightarrow \quad 0 = u_x(L,t) = F'(L)\,G(t)$

$\Rightarrow \quad F'(L) = 0$

So:

$$\boxed{\begin{aligned} F'(0) &= 0 \\ F'(L) &= 0 \end{aligned}} \qquad \begin{aligned} &(28.5a) \\ &(28.5b) \end{aligned}$$

Next, (28.4) + (28.1):

(28.4) $\Rightarrow \begin{cases} u_t(x,t) = F(x)\,G'(t) \\ u_{xx}(x,t) = F''(x)\,G(t) \end{cases}$

So (28.1) $\Rightarrow$

$$F(x)\,G'(t) = c^2\,F''(x)\,G(t)$$

Divide b.s. by $c^2 F(x)\,G(t)$ to get:

$$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k \quad (\text{const.})$$

$$\Rightarrow \boxed{\underset{(28.6a)}{G'(t) = kc^2 G(t)}, \qquad \underset{(28.6b)}{F''(x) = kF(x)}}$$

Consider (28.6b) + (28.5a,b) :-

Case a): $k = 0$

(28.6b) $\Rightarrow F''(x) = 0 \Rightarrow F(x) = ax + b$
$\Rightarrow F'(x) = a$

Then (28.5a) $\Rightarrow 0 = F'(0) = a \Rightarrow F(x) = b$
(28.5b) $\Rightarrow 0 = F'(L) = a$ — nothing new

With $k = 0$, (28.6a) $\Rightarrow G'(t) = 0 \Rightarrow G(t) = c$

Combining, we have the solution

$$u(x,t) = c.b = A_0 \quad (\text{arb. const.})$$

of (28.1) + (28.2a) + (28.2b).

Case b): $k > 0$: Set $k = \mu^2 \quad (\mu > 0)$

(28.6b) $\Rightarrow F''(x) = \mu^2 F(x)$

$$\Rightarrow \quad F(x) = a\cosh(\mu x) + b\sinh(\mu x)$$

$$\Rightarrow F'(x) = \mu a\sinh(\mu x) + \mu b\cosh(\mu x)$$

Then (28.5a) $\Rightarrow \quad 0 = 0 + \mu b \Rightarrow b = 0$

$$\Rightarrow F(x) = a\cosh(\mu x)$$

(28.5b) $\Rightarrow \quad 0 = \mu a\sinh(\mu L) \Rightarrow a = 0$

$\Rightarrow F(x) \equiv 0 \Rightarrow u(x,t) = 0$ trivial.

<u>**Case c): $k<0$: Set $k = -p^2$ $\quad (p>0)$**</u>

(28.6b) $\Rightarrow \quad F''(x) = -p^2 F(x)$

$$\Rightarrow \quad F(x) = a\cos(px) + b\sin(px)$$

$$\Rightarrow \quad F'(x) = -ap\sin(px) + bp\cos(px)$$

Then (28.5a) $\Rightarrow \quad 0 = 0 + bp \Rightarrow b = 0$

$$\Rightarrow F(x) = a\cos(px)$$

(28.5b) $\Rightarrow \quad 0 = -ap\sin(pL)$

$\Rightarrow a = 0 \quad (\Rightarrow F(x) = 0 \Rightarrow u(x,t) = 0$ trivial)

OR

$pL = n\pi, \qquad n = 1$ or 2 or $3 \dots$

and $F(x) = a\cos\left(\frac{n\pi x}{L}\right)$

With $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$, (28.6a) gives

$$G'(t) = -c^2\left(\frac{n\pi}{L}\right)^2 G(t)$$

$$\Rightarrow \quad G(t) = D e^{-\left(\frac{n\pi c}{L}\right)^2 t}$$

Combining, we have the solution

$$u(x,t) = a.D\, e^{-\left(\frac{n\pi c}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

for $n = 1$ or 2 or $3 \ldots$

Thus we get an infinite sequence of solutions

$$u_0(x,t) = A_0, \qquad u_n(x,t) = A_n\, e^{-\left(\frac{n\pi c}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

arb. constants

$n = 1, 2, 3 \ldots$ (28.7)

Following Fourier, assume general solution of (28.1) + (28.2a) + (28.2b) is obtained by superposition:

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\, e^{-\left(\frac{n\pi c}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right) \qquad (28.8)}$$

Now impose the remaining condition (28.3) to fix A_0, A_n and hence determine $u(x,t)$:

(28.3) $\Rightarrow$

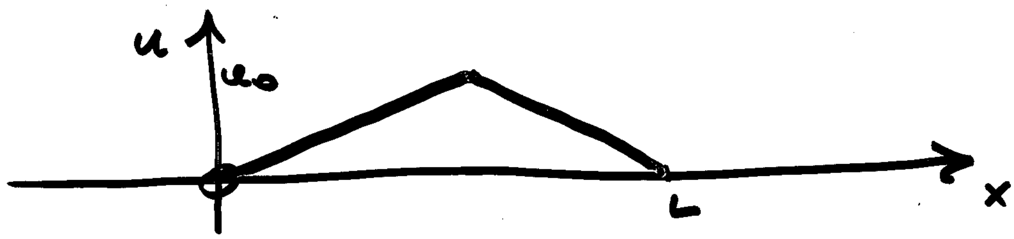
$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

Fourier half-range cosine series for $f(x)$, $0 < x < L$.

$$\Rightarrow \boxed{\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad (28.9) \\ & n = 1, 2, \dots \end{aligned}}$$

Given $f(x)$, we evaluate A_0, A_n, & sub. in (28.8). Note the rapidly decreasing exponential factors help convergence – usually only need to consider a small no. of terms to get a good approx. solution.

EX: $$f(x) = \begin{cases} \frac{2u_0}{L}x, & 0 < x < \frac{L}{2} \\ \frac{2u_0}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$



Then $$A_0 = \frac{2u_0}{L^2}\left\{\int_0^{\frac{L}{2}} x\,dx + \int_{\frac{L}{2}}^{L}(L-x)\,dx\right\}$$

$$= \frac{u_0}{2} \quad \text{Check!}$$

$$A_n = \frac{4u_0}{L^2}\left\{\int_0^{\frac{L}{2}} x\cos\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L}(L-x)\cos\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{4u_0}{n^2\pi^2}\Big(\underbrace{2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1}_{=\;0,\,-4,\,0,\,0,\,0,\,-4,\,0,\,\ldots}\Big) \quad \text{Check!}$$

$$n = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots$$

So $$u(x,t) = \tfrac{1}{2}u_0 - \frac{16u_0}{\pi^2}\left[\frac{1}{4}e^{-\left(\frac{2\pi c}{L}\right)^2 t}\cos\left(\frac{2\pi x}{L}\right) + \frac{1}{36}e^{-\left(\frac{6\pi c}{L}\right)^2 t}\cos\left(\frac{6\pi x}{L}\right) + \vdots\right]$$

Note: As $t \to \infty$, $u(x,t) \to \frac{1}{2} u_0$

For general case, (28.8), $u(x,t) \to A_0$

Physical meaning:

$$L\rho\sigma u(x,t) \to \underbrace{L\rho\sigma A_0}_{\substack{\text{final heat} \\ \text{content} \\ \text{at temp. } A_0}} = \underbrace{\int_0^L \rho\sigma f(x)\,dx}_{\substack{\text{initial heat} \\ \text{content} \\ \text{with temp. distbn.} \\ f(x)}}$$

— heat energy conserved.

Summary:

1) Fourier's Method (Separation of Variables and Superposition) applies also to 1-D Heat Equation.

2) Be able to reproduce (28.1) → (28.8)

3) Be able to do examples for 1-D wave and heat equations.

~~K §11.3, 11.5~~ K §12.3, 12.9