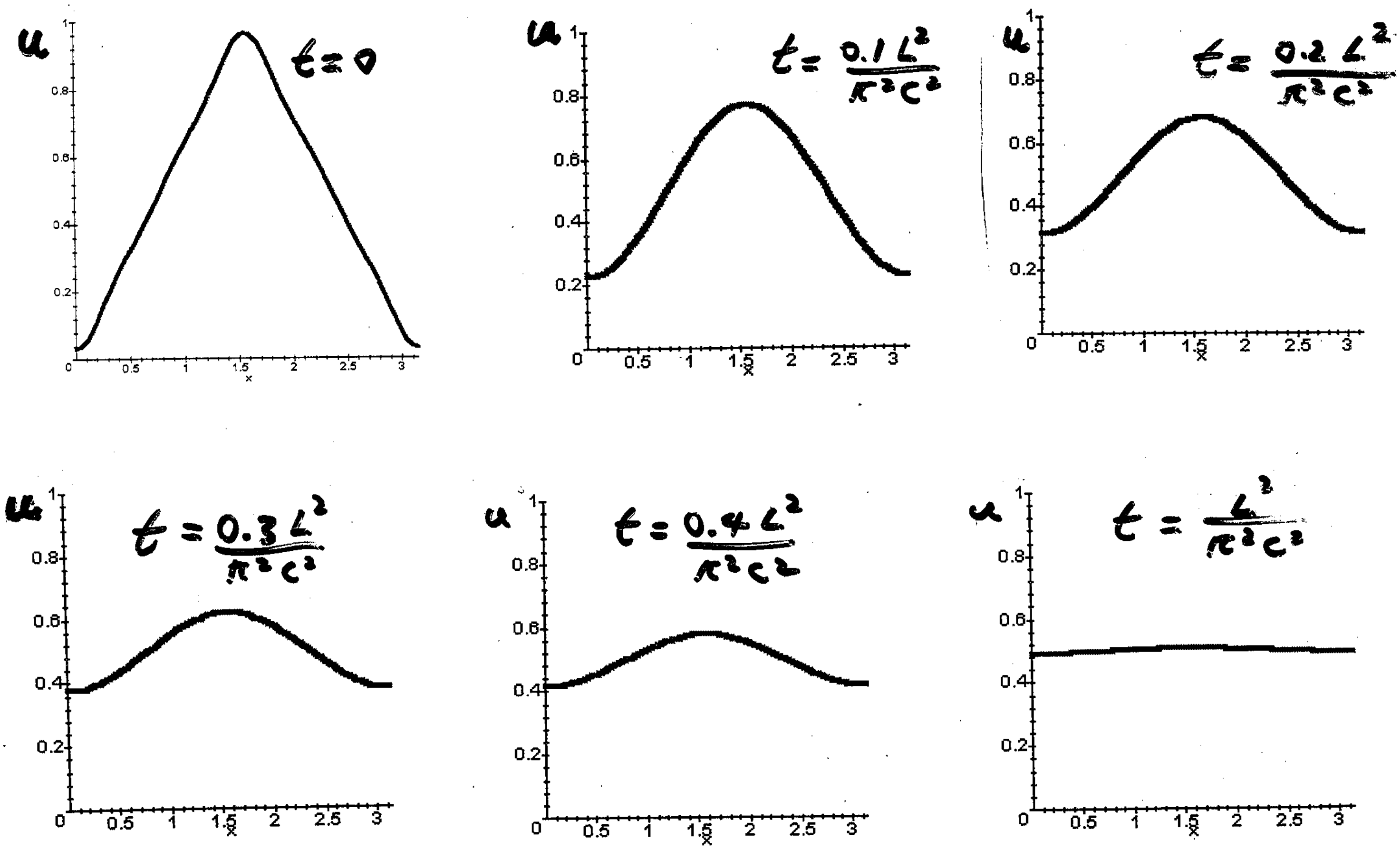
MATH2100 Lec. 29 (= MATH2011 Lec. 11)

From EX: on p. 28.11:

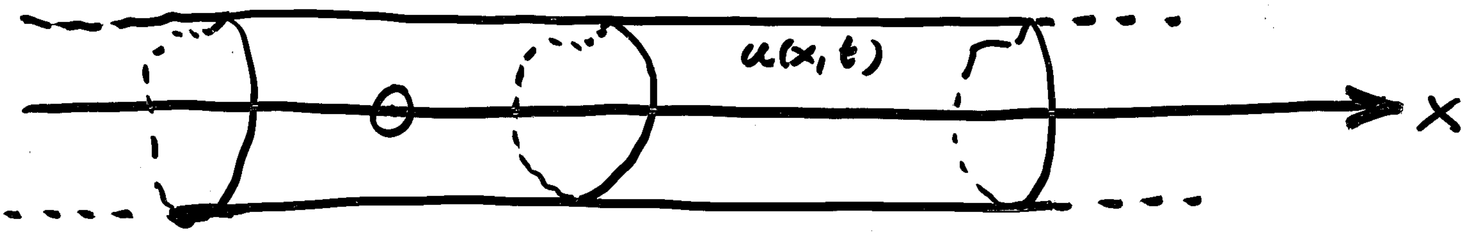
$$u(x,t) \approx \tfrac{1}{2}u_0 - \frac{16u_0}{\pi^2}\Big[\tfrac{1}{4}e^{-\left(\frac{2\pi c}{L}\right)^2 t}\cos\left(\frac{2\pi x}{L}\right) + \tfrac{1}{36}e^{-\left(\frac{6\pi c}{L}\right)^2 t}\cos\left(\frac{6\pi x}{L}\right) + \tfrac{1}{100}e^{-\left(\frac{10\pi c}{L}\right)^2 t}\cos\left(\frac{10\pi x}{L}\right)\Big]$$



$(u_0 = 1,\ c = 1,\ L = \pi)$

1-D Heat Equation on whole X-axis

Suppose we have a problem in heat conduction in a very long cylinder with insulated sides. Boundaries are effectively at $x = \pm\infty$.



We have the mathematical problem defined by:

(29.1) $$u_t(x,t) = c^2 u_{xx}(x,t) \qquad -\infty < x < \infty,\ t > 0$$

(29.2) IC: $$u(x,0) = f(x) \qquad -\infty < x < \infty$$

IVP for 1-D Heat Equation on whole X-axis.
Find $u(x,t)$, $-\infty < x < \infty$, $t > 0$

Fourier's Method not useful here.
Instead we make use of a very special solution of (29.1):-

$$G(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-x^2/4c^2t} \quad , \quad -\infty < x < \infty, \; t > 0$$

Green's fn for 1-D heat eqn?

In Assignment 8, you showed this is indeed a solution of (29.1).

Properties of $G(x,t)$:

1) $$\lim_{t \to 0_+} G(x,t) = 0 \quad , \quad \text{if} \quad x \neq 0 \qquad (29.3)$$

(because $$\lim_{t \to 0_+} \frac{e^{-\frac{A}{t}}}{\sqrt{t}} = 0 \qquad (A>0))$$

2) $$\lim_{t \to 0_+} G(0,t) = \lim_{t \to 0_+} \frac{1}{\sqrt{4\pi c^2 t}} = +\infty \qquad (29.4)$$

3) $$\int_{-\infty}^{\infty} G(x,t)\,dx = 1 \quad \text{for every} \quad t > 0 \qquad (29.5)$$

<u>Pf:</u> Consider LHS: $$\frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-x^2/4c^2t}\,dx$$

Put $$u = \frac{x}{\sqrt{4c^2t}}$$

$$dx = \sqrt{4c^2t}\,du$$

$$x = \infty \longleftrightarrow u = \infty$$
$$x = -\infty \longleftrightarrow u = -\infty$$

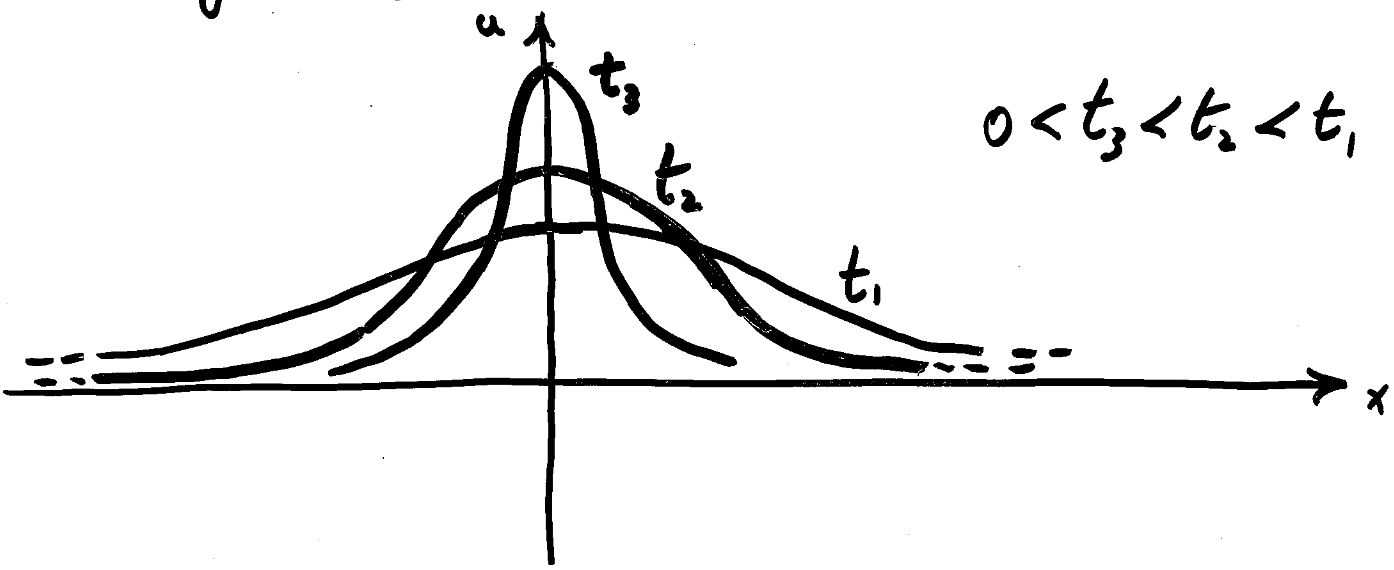
Then $$LHS = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{4c^2t}\,du$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du$$

But $$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

Hence required result.

At any $t>0$, $G(x,t)$ is a Gaussian:



Area under each curve is 1.

As $t \to 0_+$, approach an <u>infinite spike</u> at $x=0$.

By 3), amount of heat energy in system at any time is

$$H(t) = \int_{-\infty}^{\infty} \rho\sigma\, G(x,t)\,dx = \rho\sigma \quad \text{(const.)}$$

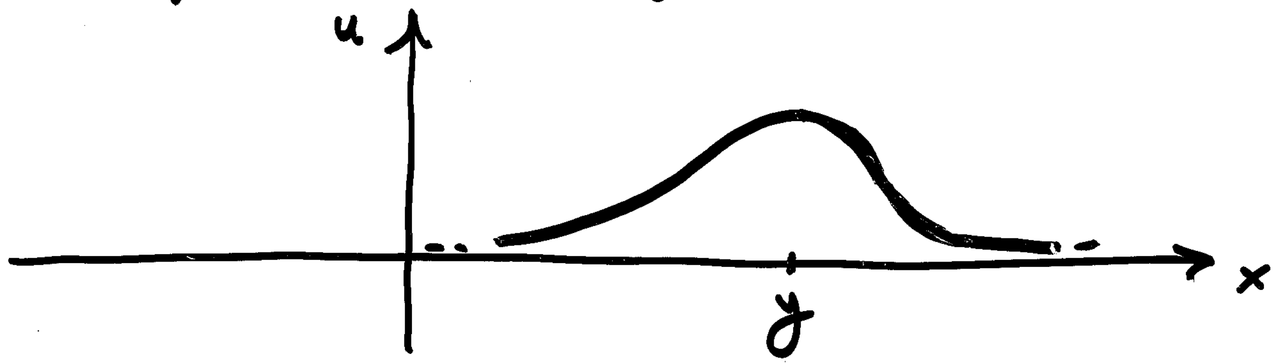
As $t \to 0_+$, all this heat energy is localised at $x=0$.

Can say: $G(x,t)$ is the temperature distribution at $t>0$ resulting from injection of amount of heat $p\sigma$ into system at zero temperature, at $x=0$ at $t=0$.

<u>Next</u>: See $G(x-y,t) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-(x-y)^2/4c^2 t}$ is also a solution of the PDE (29.1), for any <u>constant</u> y.

$$G_t(x-y,t) = c^2 G_{xx}(x-y,t) \qquad (29.6)$$

See $G(x-y,t)$ is temp. distbn at $t>0$ resulting from injection of amount of heat $p\sigma$ into system at $x=y$ at $t=0$.



<u>Next</u>: See $\alpha_1 G(x-y_1,t) + \alpha_2 G(x-y_2,t) + \cdots$ is also a solution. [At $t=0$, inject heat $\alpha_1 p\sigma$ at y_1, $\alpha_2 p\sigma$ at y_2, ...]

Claim that solution of PDE (29.1) + IC (29.2) is:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, f(y)\, dy \qquad (29.7)$$

[Think of as resulting from injection of heat $\rho c f(y)\delta y$ in $(y, y+\delta y)$, all along x-axis, just right to establish initial temp. distb^n]

In any case, can check directly that (29.7) does satisfy (29.1) + (29.2):

Firstly, check (29.1):

$$u_t(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy = \int_{-\infty}^{\infty} G_t(x-y,t) f(y)\, dy$$

$$u_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy = \int_{-\infty}^{\infty} G_{xx}(x-y,t) f(y)\, dy$$

Then

$$u_t(x,t) - c^2 u_{xx}(x,t) = \int_{-\infty}^{\infty} \left[G_t(x-y,t) - c^2 G_{xx}(x-y,t)\right] f(y)\,dy$$

$$= 0 \quad \text{by (29.6)}$$

Next, check (29.2):

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2 t} f(y)\,dy$$

Change variable of integration:

$(y \to w)$ $\quad w = (x-y)/\sqrt{4c^2 t} \iff y = x - \sqrt{4c^2 t}\, w$

$$dy = -\sqrt{4c^2 t}\, dw$$

$w = +\infty \iff y = -\infty \qquad w = -\infty \iff y = +\infty$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{+\infty}^{-\infty} e^{-w^2} f(x - \sqrt{4c^2 t}\, w)\left(-\sqrt{4c^2 t}\right) dw$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-w^2} f(x - \sqrt{4c^2 t}\, w)\, dw$$

Then, as $t \to 0_+$,

$$u(x,t) \to f(x)\,\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,du = f(x)$$

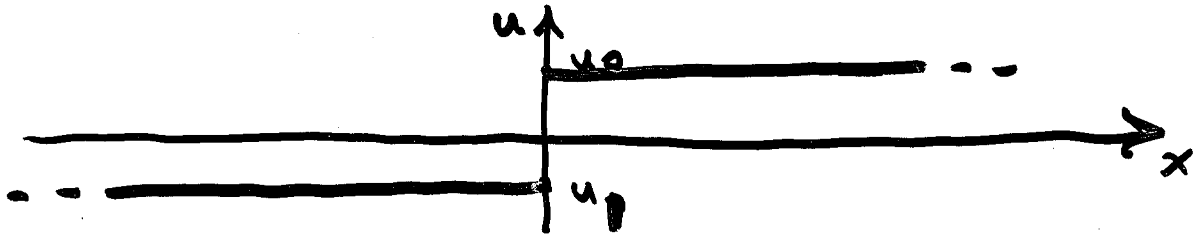
as required.

So: Solution of PDE (29.1) + IC (29.2) is:-

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,f(y)\,dy = \frac{1}{\sqrt{4\pi c^2 t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2t} f(y)\,dy} \qquad (29.8)$$

EX:

$$f(x) = \begin{cases} u_0 \text{ (const.)} & x > 0 \\ u_1 \text{ (const.)} & x < 0 \end{cases}$$



Solution is, acc. to (29.8):–

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \left\{ u_0 \int_0^\infty e^{-(x-y)^2/4c^2t}\, dy + u_1 \int_{-\infty}^0 e^{-(x-y)^2/4c^2t}\, dy \right\}$$

Change variable of integration:

$$v = \frac{(x-y)}{\sqrt{4c^2t}} \Leftrightarrow y = x - \sqrt{4c^2t}\, v$$

$$dy = -\sqrt{4c^2t}\, dv$$

$$y = \infty \Leftrightarrow v = -\infty \qquad y = -\infty \Leftrightarrow v = +\infty$$

$$y = 0 \Leftrightarrow v = x/\sqrt{4c^2t}$$

$\Rightarrow$

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \left\{ u_0(-\sqrt{4c^2t}) \int_{x/\sqrt{4c^2t}}^{-\infty} e^{-v^2} dv + u_1(-\sqrt{4c^2t}) \int_{\infty}^{x/\sqrt{4c^2t}} e^{-v^2} dv \right\}$$

$$= u_0 \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4c^2t}} e^{-v^2} dv + u_1 \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4c^2t}}^{\infty} e^{-v^2} dv$$

Now $$\int_{-\infty}^{x/\sqrt{\;}} \cdots = \int_{-\infty}^{0} \cdots + \int_{0}^{x/\sqrt{\;}} \cdots$$

$$\int_{x/\sqrt{\;}}^{\infty} \cdots = \int_{0}^{\infty} \cdots + \int_{x/\sqrt{\;}}^{0} \cdots = \int_{0}^{\infty} \cdots - \int_{0}^{x/\sqrt{\;}} \cdots$$

So $$u(x,t) = u_0 \left\{ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-v^2}\,dv + \frac{1}{\sqrt{\pi}} \int_{0}^{x/\sqrt{4c^2t}} e^{-v^2}\,dv \right\}$$

$$+ u_1 \left\{ \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-v^2}\,dv - \frac{1}{\sqrt{\pi}} \int_{0}^{x/\sqrt{4c^2t}} e^{-v^2}\,dv \right\}$$

Now $$\int_{-\infty}^{0} e^{-v^2}\,dv = \int_{0}^{\infty} e^{-v^2}\,dv = \frac{1}{2}\int_{-\infty} e^{-v^2}\,dv = \frac{\sqrt{\pi}}{2}$$

So $$u(x,t) = \tfrac{1}{2}(u_0 + u_1) + \tfrac{1}{2}(u_0 - u_1)\left\{ \frac{2}{\sqrt{\pi}} \int_{0}^{x/\sqrt{4c^2t}} e^{-v^2}\,dv \right\}$$

We define a new function (as on p. 26.10, 11, 12)

$$\boxed{\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-v^2}\,dv \qquad (29.9)}$$

— the error function

Then our solution is:

$$u(x,t) = \tfrac{1}{2}(u_0+u_1) + \tfrac{1}{2}(u_0-u_1)\,\mathrm{erf}\left(\frac{x}{\sqrt{4c^2t}}\right)$$

<u>Summary</u>:

1) Know the form of $G(x-y,t)$ Green's Function for 1-D Heat Equ^n.
Check satisfies 1-D Heat Equ^n !!

2) Know form of solution (29.7) of 1-D Heat Equ^n (29.1) + IC (29.2). Be able to check it <u>is</u> solution.

3) Understand manipulations in final example.

4) Know definition of $\mathrm{erf}(z)$

K p. ~~612~~ 564 EX. 1