Lec. 2

The ODEs

$$y'(t) = 3\sin(t)\,y(t) + 7e^t$$
$$y''(x) + 6e^{x^2}y'(x) - 3y(x) = 4x$$
$$\ddot{x}(t) + \cos(t^3)\,x(t) = 4$$

$$\begin{cases} \cos(7t)\,y_1'(t) = 6y_1(t) + 4t^3 y_2(t) + e^{3t} \\ \sin(4t)\,y_2'(t) = 4y_1(t) + 2t^2 y_2(t) \end{cases}$$

are all <u>linear</u> - unknowns and derivative appear <u>linearly</u>.

The ODEs

$$y'(t) = 3\sin(y(t)) + 7e^t$$
$$[y''(x)]^2 + 6y'(x) - 3y(x)^4 = 4x$$
$$\ddot{x}(t) + \cos(x(t)^3)\,x(t) = 4$$

$$\begin{cases} \cos(3t)\,y_1'(t) = 6y_1(t) + 4t^3 y_1(t)y_2(t) + e^{3t} \\ \sin(4t)\,y_1(t)y_2'(t) = 4y_1(t) + 2t^2(y_2(t))^2 \end{cases}$$

are all nonlinear - unknowns and/or derivatives appear in nonlinear combinations. Much harder - often impossible - to solve.

The ODEs

$$y'(t) = 3\sin(t)\, y(t)$$
$$y''(x) + 6e^{x^2} y'(x) - 3y(x) = 0$$
$$\ddot{x}(t) + \cos(t^3)\, x(t) = 0$$

$$\begin{cases} \cos(3t)\, y_1'(t) = 6y_1(t) + 4t^3 y_2(t) \\ \sin(4t)\, y_2'(t) = 4y_1(t) + 2t^2 y_2(t) \end{cases}$$

are all <u>homogeneous</u> as well as <u>linear</u>.

[Test: When replace $y(t)$ by $\lambda y(t)$
$y(x)$ by $\lambda y(x)$
$x(t)$ by $\lambda x(t)$
$y_1(t), y_2(t)$ by $\lambda y_1(t), \lambda y_2(t)$

where λ is an arbitrary constant, equations are unchanged.

EX: (*)
$$\begin{cases} \cos(3t)\, y_1'(t) = 6 y_1(t) + 4t^3 y_2(t) \\ \sin(4t)\, y_2'(t) = 4 y_1(t) + 2t^2 y_2(t) \end{cases}$$

$$\rightarrow \begin{cases} \cos(3t)\, \lambda y_1'(t) = 6\lambda y_1(t) + 4t^3 \lambda y_2(t) \\ \sin(4t)\, \lambda y_2'(t) = 4\lambda y_1(t) + 2t^2 \lambda y_2(t) \end{cases}$$

Homogeneous, linear DEs obey

<u>SUPERPOSITION PRINCIPLE</u>:

can take linear combinations of known solutions to form new solutions

In above example, if $\begin{pmatrix} y_{1A}(t) \\ y_{2A}(t) \end{pmatrix}$ is one solution of (*), and $\begin{pmatrix} y_{1B}(t) \\ y_{2B}(t) \end{pmatrix}$ is another solution, then

$$\begin{pmatrix} y_{1C}(t) \\ y_{2C}(t) \end{pmatrix} = \alpha \begin{pmatrix} y_{1A}(t) \\ y_{2A}(t) \end{pmatrix} + \beta \begin{pmatrix} y_{1B}(t) \\ y_{2B}(t) \end{pmatrix}$$

is also a solution for arbitrary constants α, β.

The ODEs

$$y'(t) = 3\sin(t)\, y(t) + 4$$
$$y''(x) + 6e^{x^2} y'(x) - 3y(x) = \ln(x)$$
$$\ddot{x}(t) + \cos(t^3)\, x(t) = 5e^t$$

$$\begin{cases} \cos(3t)\, y_1'(t) = 6y_1(t) + 4t^3 y_2(t) + e^{t^3} \\ \sin(4t)\, y_2'(t) = 4y_1(t) + 2t^2 y_2(t) \end{cases}$$

are all linear but $\begin{cases} \underline{\text{inhomogeneous}} \\ \underline{\text{nonhomogeneous}} \end{cases}$

Check!

Superposition principle does **NOT** apply.

We are going to spend some time looking at <u>systems of two coupled 1st order ODEs</u>.

Consider 2nd order linear ODE:

$$1\,y''(t) + k(t)\,y'(t) + q(t)\,y(t) = \underbrace{r(t)}_{\text{forcing term}}$$

coefficients

Put $y_1(t) = y(t)$, $y_2(t) = y'(t)$

Then

$$y_1'(t) = y'(t) = y_2(t)$$

$$y_2'(t) = y''(t) = -k(t)\,y'(t) - q(t)\,y(t) + r(t)$$

$$= -k(t)\,y_2(t) - q(t)\,y_1(t) + r(t)$$

Now have two coupled 1st order ODEs:

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = -k(t)\,y_2(t) - q(t)\,y_1(t) + r(t) \end{cases}$$

In matrix notation:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -k(t) \end{bmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

or

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = A(t) \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

or

$$\boxed{\underset{\sim}{y}'(t) = A(t)\,\underset{\sim}{y}(t) + \underset{\sim}{r}(t)}$$

If $\underset{\sim}{r}(t) = \underset{\sim}{0}$, system is homogeneous

If $\underset{\sim}{r}(t) \neq \underset{\sim}{0}$, system is inhomogeneous

Let's look at case when

1) $r(t) = 0$ ($\Rightarrow \underset{\sim}{r}(t) = \underset{\sim}{0}$) homogeneous

2) constant coefficients k, q

$$y''(t) + k\,y'(t) + q\,y(t) = 0 \qquad (2.1)$$

$\Rightarrow$

$$(2.2) \quad \underset{\sim}{y}'(t) = A\,\underset{\sim}{y}(t) \quad , \quad A = \begin{bmatrix} 0 & 1 \\ -q & -k \end{bmatrix}$$

(2.2) is system of 2 coupled, linear, homogeneous 1st order ODEs, with constant coefficients.

We know how to find general solution of (2.1), so must be able to find general solution of (2.2)!

Also, we know how to find the <u>unique</u> solution of (2.1) with <u>initial</u> <u>conditions</u> (<u>ICs</u>)

$$y(0) = a_1 \quad , \quad y'(0) = a_2 \qquad (2.3)$$

so we must know how to solve (2.2) with the ICs

$$y_1(0)\,[= y(0)] = a_1$$
$$y_2(0)\,[= y'(0)] = a_2$$

i.e.

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad (2.4)$$

i.e.

$$\underset{\sim}{y}(0) = \underset{\sim}{a}$$

let's <u>see</u> how it works!

To solve (2.2), try

$$\underset{\sim}{y}(t) = \underset{\sim}{x}\, e^{\lambda t} \qquad (2.5)$$

$\underset{\sim}{x} = \begin{pmatrix} u \\ v \end{pmatrix}$ constant vector, λ constant.

Then $$\underset{\sim}{y}'(t) = \lambda \underset{\sim}{x}\, e^{\lambda t} \qquad (2.6)$$

Sub. (2.5), (2.6) in (2.2) :—

$$\lambda \underset{\sim}{x} e^{\lambda t} = A \underset{\sim}{x} e^{\lambda t}$$

$$\Rightarrow \quad \boxed{A\underset{\sim}{x} = \lambda \underset{\sim}{x}} \qquad (2.7)$$

<u>eigenvalue equation for A</u>

If λ_1, λ_2 are (distinct) eigenvalues;

and $\underset{\sim}{x}^{(1)}$, $\underset{\sim}{x}^{(2)}$ are corresp. eigenvectors, then
$\underset{\sim}{x}^{(1)} e^{\lambda_1 t}$ and $\underset{\sim}{x}^{(2)} e^{\lambda_2 t}$ are ~~both~~ each

solutions of (2.2). And then

$$\underset{\sim}{y}(t) = c_1 \underset{\sim}{x}^{(1)} e^{\lambda_1 t} + c_2 \underset{\sim}{x}^{(2)} e^{\lambda_2 t} \quad (2.8)$$

is also a solution (superposition principle!) for arbitrary constants c_1, c_2. In (2.8), we have the general solution of (2.2).

Use ICs to fix c_1, c_2 and make solution unique.

EX: $y''(t) + 2y'(t) - 15y(t) = 0$ (2.9)

Two ways to solve now.

a) <u>Directly</u>: (old way!)

Try $y(t) = e^{\lambda t}$

$\Rightarrow$ $y'(t) = \lambda e^{\lambda t}$, $y''(t) = \lambda^2 e^{\lambda t}$

Sub. in (2.9):-

$$\lambda^2 \cancel{e^{\lambda t}} + 2\lambda \cancel{e^{\lambda t}} - 15 \cancel{e^{\lambda t}} = 0$$

$\Rightarrow$ $\lambda^2 + 2\lambda - 15 = 0$ characteristic quadratic

$\Rightarrow$ $(\lambda+5)(\lambda-3) = 0$

$\Rightarrow$ $\lambda_1 = -5$, $\lambda_2 = 3$ roots of quadratic

Then $y(t) = e^{-5t}$ and $y(t) = e^{3t}$ are each solutions of (2.9)

General solution of (2.9) is

$$y(t) = \alpha e^{3t} + \beta e^{-5t}$$

α, β arbitrary constants.

If we have ICs

$$y(0) = -1 \quad , \quad y'(0) = 13$$

then need

$$\alpha + \beta = -1 \quad , \quad 3\alpha - 5\beta = 13$$

$$\Rightarrow \alpha = 1 \quad , \quad \beta = -2$$

$$\Rightarrow \underline{y(t) = e^{3t} - 2e^{-5t}}$$

b) <u>As a system</u>: (new way!)

$$y''(t) + 2y'(t) - 15y(t) = 0$$

Let $y_1(t) = y(t)$, $y_2(t) = y'(t)$

Then $y_1' = y' = y_2$ $\quad y_2' = y''$

$$= 15y - 2y'$$

$$= 15y_1 - 2y_2$$

So $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 15 & -2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

or

$$\underset{\sim}{y}'(t) = A\,\underset{\sim}{y}(t), \quad A = \begin{bmatrix} 0 & 1 \\ 15 & -2 \end{bmatrix}$$

Try $\underset{\sim}{y}(t) = \underset{\sim}{x}\, e^{\lambda t}$ $\left(\Rightarrow \underset{\sim}{y}'(t) = \lambda \underset{\sim}{x} e^{\lambda t}\right)$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} e^{\lambda t}$$

$$\Rightarrow \quad \lambda \underset{\sim}{x} \cancel{e^{\lambda t}} = A \underset{\sim}{x} \cancel{e^{\lambda t}}$$

$$\Rightarrow \quad A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\Leftrightarrow \quad (A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I\underset{\sim}{x} = \underset{\sim}{x}$$

Eigenvalue condition: (for nontrivial $\underset{\sim}{x}$)

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \quad \begin{vmatrix} 0-\lambda & 1 \\ 15 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-\lambda)(-\lambda-2) - 15 = 0$$

$$\Rightarrow \quad \lambda^2 + 2\lambda - 15 = 0 \quad \text{(charac. quadratic!)}$$

Eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -5$ 3.4 = 2.14

Finding e'vectors: $\underline{\lambda_1 = 3}$:

$$(A - 3I)\underset{\sim}{x} = \underline{0}$$

$$\Rightarrow \quad \begin{bmatrix} 0-3 & 1 \\ 15 & -2-3 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} -3u + v &= 0 \\ 15u - 5v &= 0 \end{aligned}\right\} \Rightarrow v = 3u$$

Choosing $u = 1$, e'vector is $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\underline{\lambda_2 = -5}$:

$$(A + 5I)\underset{\sim}{x} = \underline{0}$$

$$\Rightarrow \quad \begin{bmatrix} 5 & 1 \\ 15 & 3 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} 5u + v &= 0 \\ 15u + 3v &= 0 \end{aligned}\right\} \Rightarrow v = -5u$$

Choosing $u = 1$, e'vector is $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

So now know that $\underline{x}^{(1)} e^{3t}$ and $\underline{x}^{(2)} e^{-5t}$ are solutions.

Then general solution is

$$\underset{\sim}{y}(t) = \alpha \underset{\sim}{x}^{(1)} e^{3t} + \beta \underset{\sim}{x}^{(2)} e^{-5t}$$

i.e. $$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + \beta \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-5t}$$

[Recall $y(t) = y_1(t)$ — top component
$= \alpha e^{3t} + \beta e^{5t}$ — as before

Also $\underline{y_2(t) = y'(t) = 3\alpha e^{3t} - 5\beta e^{-5t}}$

With IC, $y(0) = -1$ $(\Rightarrow y_1(0) = -1)$
$y'(0) = 13$ $(\Rightarrow y_2(0) = 13)$

we get

$$\underset{\sim}{y}(0) = \begin{pmatrix} -1 \\ 13 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$\Rightarrow \alpha = 1, \ \beta = -2$ as before,

So get unique solution:

$$\underset{\sim}{y}(t) = \underset{\sim}{x}^{(1)} e^{3t} - 2 \underset{\sim}{x}^{(2)} e^{-5t}$$

Summary:

1. Know meaning of linear, nonlinear homogeneous, inhomogeneous.
2. Understand idea of <u>superposition principle</u> for linear, homogeneous DEs.

3. Know how to solve 2nd order, linear ODE with constant coefficients in two ways:
   a) Directly
   b) By converting to a system, then finding e'values and e'vectors.

~~K: §1.6, §2.1, p. 124, p. 160, §3.0, §3.1~~

K: 1.5, 3.1, 4.0, 4.1, 4.2, 7.1, 7.2, 8.1