MATH2100 Lec. 30 (= MATH2011 Lec. 12)

Our solution of

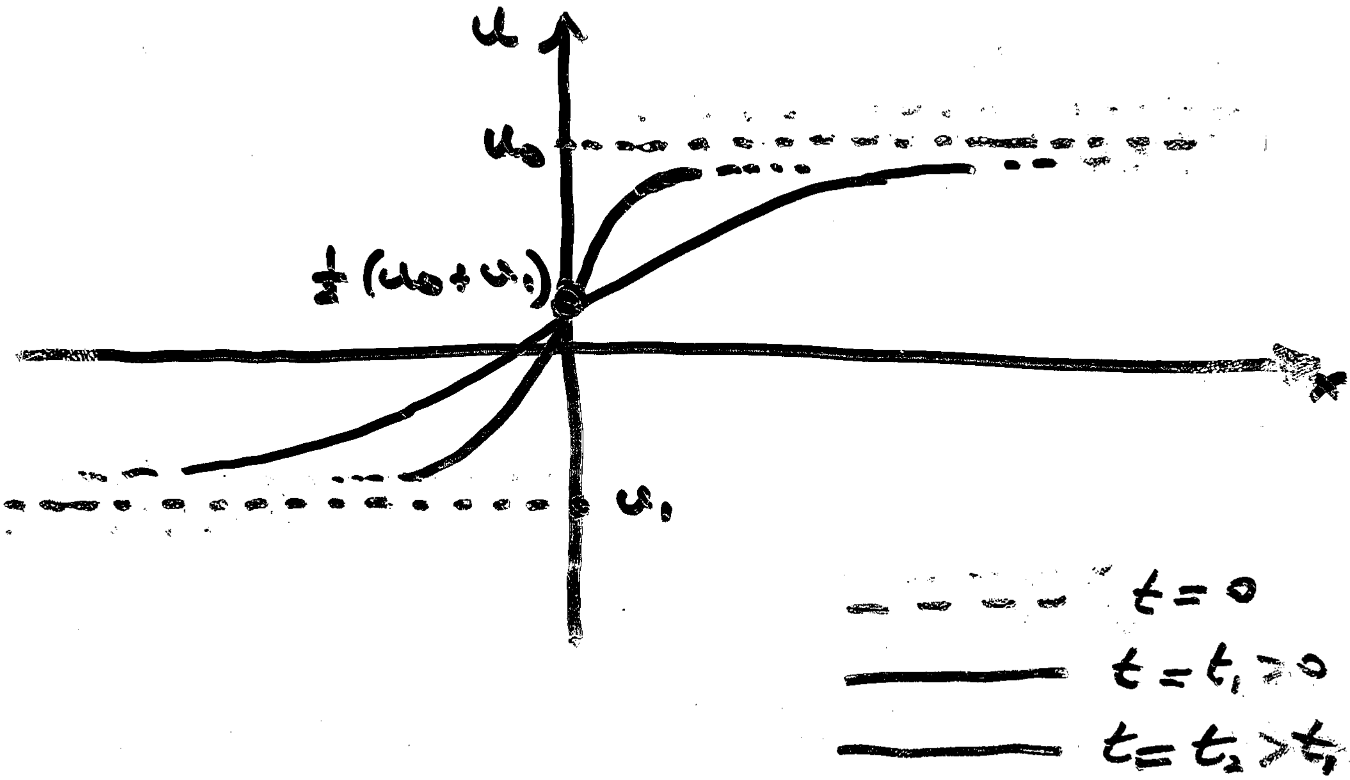
$$u_t(x,t) = c^2 u_{xx}(x,t)$$

$$-\infty < x < \infty, \quad t > 0$$

$$u(x,0) = f(x) = \begin{cases} u_0, & x > 0 \\ u_1, & x < 0 \end{cases}$$

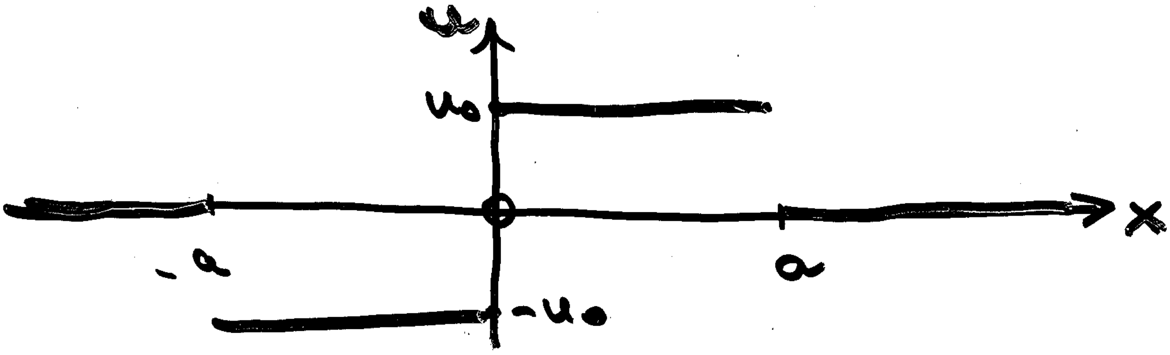
is

$$u(x,t) = \tfrac{1}{2}(u_0 + u_1) + \tfrac{1}{2}(u_0 - u_1)\,\mathrm{erf}\left(\frac{x}{\sqrt{4c^2t}}\right)$$



One more example: $u_t(x,t) = c^2 u_{xx}(x,t)$ $\quad -\infty < x < \infty,\ t > 0$

IC: $u(x,0) = f(x) = \begin{cases} u_0, & 0 < x < a \\ -u_0, & -a < x < 0 \\ 0, & |x| > a \end{cases}$



Solution is, acc. to (29.8):–

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2t} f(y)\, dy$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^a e^{-(x-y)^2/4c^2t} u_0\, dy - \int_{-a}^{0} e^{-(x-y)^2/4c^2t} u_0\, dy \right\}$$

In each integral: $\quad y \to v = (x-y)/\sqrt{4c^2t}$

$$y = x - \sqrt{4c^2t}\, v$$

$$dy = -\sqrt{4c^2t}\, dv$$

$$y = a \Leftrightarrow v = (x-a)/\sqrt{4c^2t}$$

$$y=-a \iff v=(x+a)/\sqrt{4c^2t}$$
$$y=0 \iff v = x/\sqrt{4c^2t}$$

So

$$u(x,t)=\frac{u_0}{\sqrt{\pi}}\left\{\int_{(x+a)/\sqrt{4c^2t}}^{x/\sqrt{4c^2t}} e^{-v^2}dv - \int_{x/\sqrt{4c^2t}}^{(x-a)/\sqrt{4c^2t}} e^{-v^2}dv\right\}$$

$$= \frac{u_0}{2}\left\{2\cdot\frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4c^2t}} e^{-v^2}dv - \frac{2}{\sqrt{\pi}}\int_0^{(x+a)/\sqrt{4c^2t}} e^{-v^2}dv - \frac{2}{\sqrt{\pi}}\int_0^{(x-a)/\sqrt{4c^2t}} e^{-v^2}dv\right\}$$

$$= \tfrac{1}{2}u_0\left\{2\,\mathrm{erf}\left(\frac{x}{\sqrt{4c^2t}}\right)-\mathrm{erf}\left(\frac{x+a}{\sqrt{4c^2t}}\right)-\mathrm{erf}\left(\frac{x-a}{\sqrt{4c^2t}}\right)\right\}$$

See pictures next page (different notation: $c^2 \to D$, $u(x,t) \to c(x,t)$, $u_0 \to c_0$)

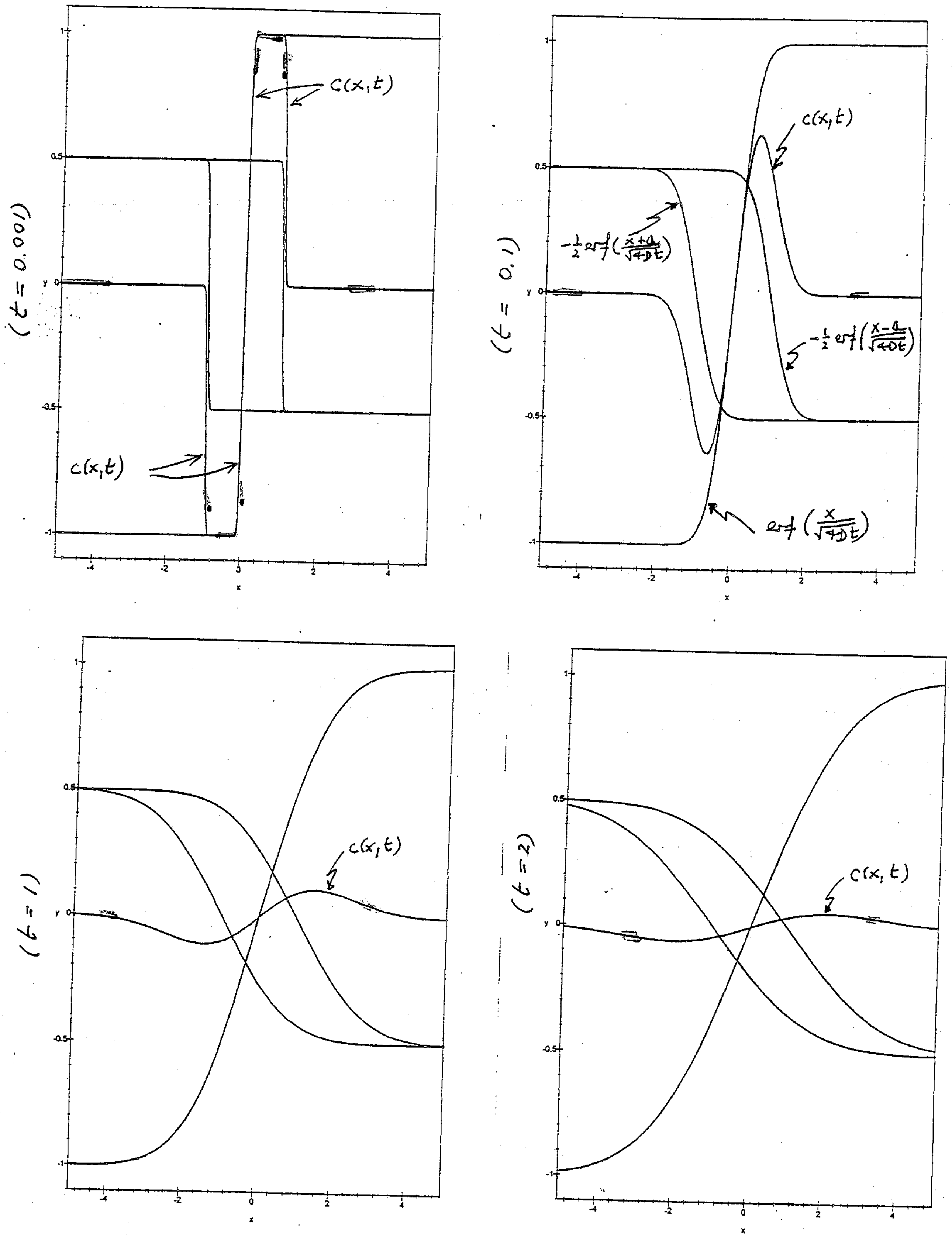
Note: Dimensions of {coeff. of thermal diffusion / thermal diffusivity}

$[c^2] = \frac{L^2}{T}$ Units could be cm²/sec, m²/hr, ..

$\Rightarrow$ If l is a characteristic length in a problem, then $\tau = l^2/c^2$ is the characteristic time for thermal diffusion to be effective over distance l

Graphs of $\operatorname{erf}\left(\frac{x}{\sqrt{4Dt}}\right)$, $-\frac{1}{2}\operatorname{erf}\left(\frac{x+a}{\sqrt{4Dt}}\right)$, $-\frac{1}{2}\operatorname{erf}\left(\frac{x-a}{\sqrt{4Dt}}\right)$ and

$$c(x,t) = \tfrac{1}{2}c_0\left[2\operatorname{erf}\left(\frac{x}{\sqrt{4Dt}}\right) - \operatorname{erf}\left(\frac{x+a}{\sqrt{4Dt}}\right) - \operatorname{erf}\left(\frac{x-a}{\sqrt{4Dt}}\right)\right]. \qquad \left\{\begin{matrix} c_0 = 1 \\ D = 1 \\ a = 1 \end{matrix}\right\}$$



We note by symmetry, solution $u(x,t)$ of previous problem satisfies $u(0,t)=0$, $t>0$

So, solution of $u_t(x,t) = c^2 u_{xx}(x,t)$ $\quad 0<x<\infty,\ t>0$

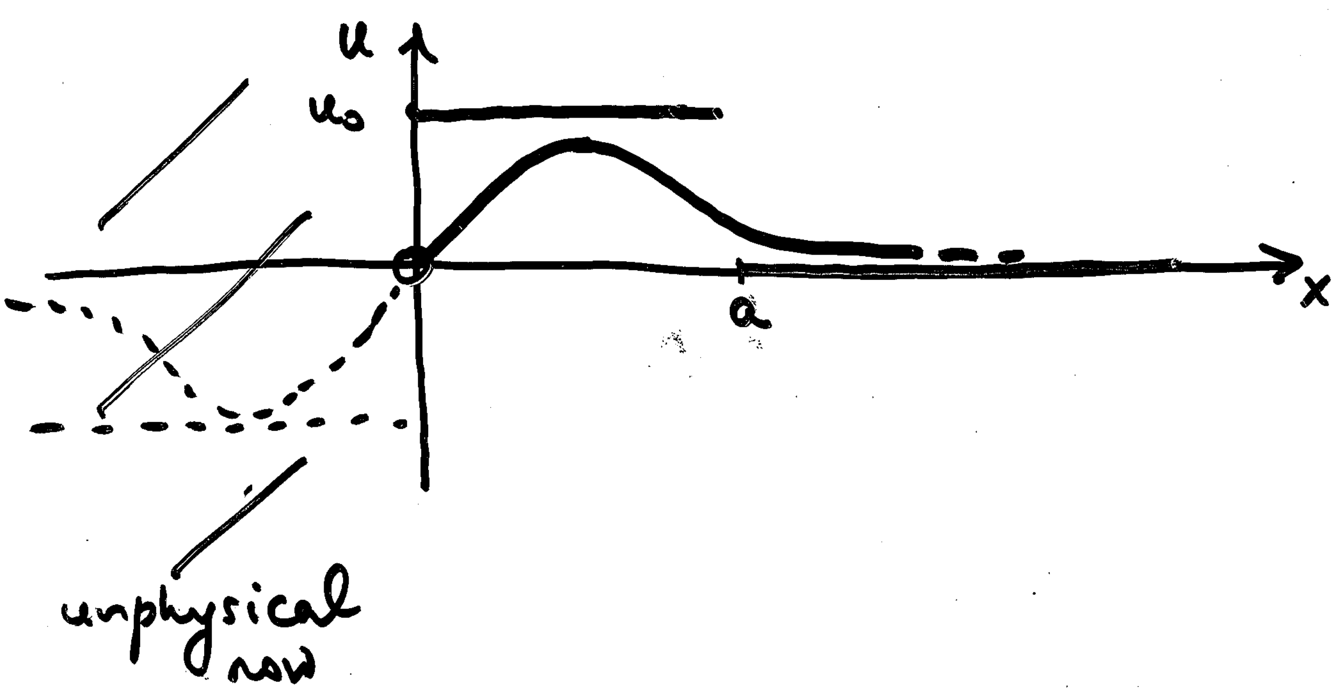
with IC: $u(x,0) = \begin{cases} u_0 & 0<x<a \\ 0 & x>a \end{cases}$

and BC: $u(0,t) = 0 \quad , \quad t>0$

is also

$$u(x,t) = \tfrac{1}{2}u_0\left\{2\,\mathrm{erf}\left(\frac{x}{\sqrt{4c^2t}}\right) - \mathrm{erf}\left(\frac{x+a}{\sqrt{4c^2t}}\right) - \mathrm{erf}\left(\frac{x-a}{\sqrt{4c^2t}}\right)\right\}$$

but now restricted to $x>0$ as well as $t>0$!!



But now see how to solve

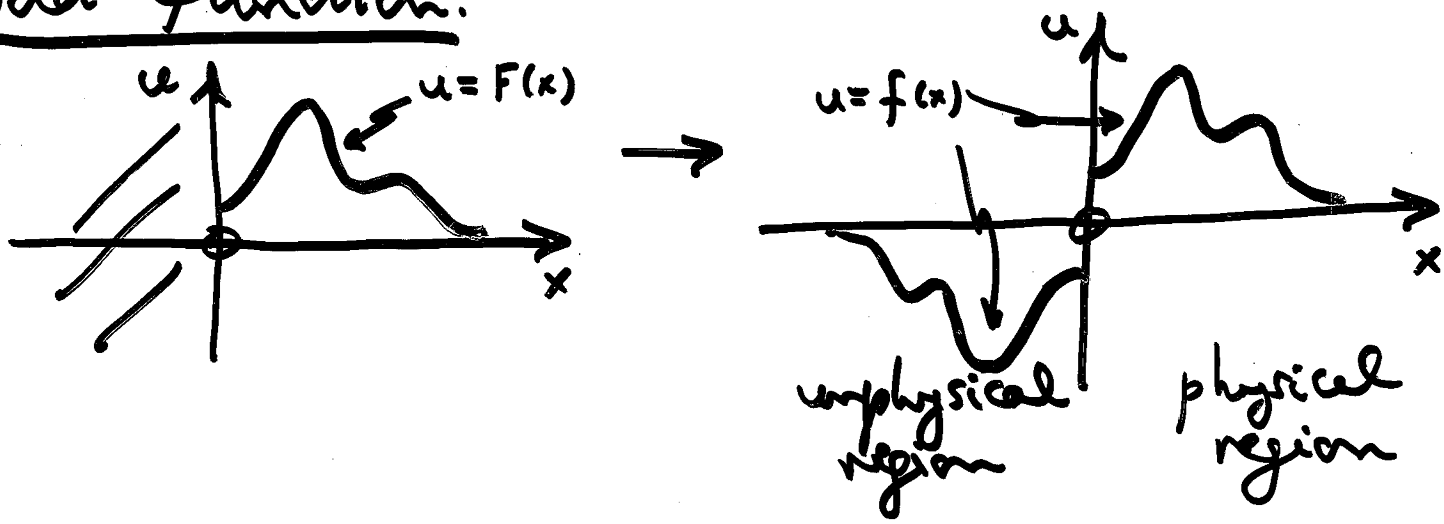
$$u_t(x,t) = c^2 u_{xx}(x,t), \qquad 0<x<\infty,\ t>0$$

with

$$\text{IC} \quad u(x,0) = F(x), \qquad 0<x<\infty$$

$$\text{BC} \quad u(0,t) = 0, \qquad t>0$$

Trick is to convert to a problem on whole x-axis, with no boundary, by extending the initial condition to an odd function.



$$F(x),\quad 0<x<\infty \quad \longrightarrow \quad f(x) = \begin{cases} F(x), & 0<x<\infty \\ -F(-x), & -\infty<x<0 \end{cases}$$

Solution of extended problem on $-\infty<x<\infty$ also provides solution of original problem on $0<x<\infty$, including BC!

This solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2t} f(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^{\infty} e^{-(x-y)^2/4c^2t} F(y)\,dy \right.$$

$$\left. - \int_{-\infty}^{0} e^{-(x-y)^2/4c^2t} F(-y)\,dy \right\}$$

$$z = -y \qquad dy = -dz$$

$$y = -\infty \Leftrightarrow z = \infty$$
$$y = 0 \Leftrightarrow z = 0$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^{\infty} e^{-(x-y)^2/4c^2t} F(y)\,dy \right.$$

$$\left. - \int_{\infty}^{0} e^{-(x+z)^2/4c^2t} F(z)\,(-dz) \right\}$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^{\infty} e^{-(x-y)^2/4c^2t} F(y)\,dy - \int_0^{\infty} e^{-(x+z)^2/4c^2t} F(z)\,dz \right\}$$

dummy $z \to y$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^{\infty} \left[e^{-(x-y)^2/4c^2t} - e^{-(x+y)^2/4c^2t} \right] F(y)\,dy \right.} \tag{30.1}$$

See $u(0,t) = \frac{1}{\sqrt{4\pi c^2 t}}\left\{ \int_0^\infty \left[e^{-y^2/4c^2t} - e^{-y^2/4c^2t} \right] F(y)\,dy \right\}$

$= 0$

Note can write solution as

$$\boxed{u(x,t) = \int_0^\infty \left[G(x-y,t) - G(x+y,t) \right] F(y)\,dy \qquad (30.2)}$$

This is called <u>Method of Images</u>

Suppose BC were $u(0,t) = u_0 \neq 0, \; t > 0$

First put $\hat{u}(x,t) = u(x,t) - u_0$

Then $\hat{u}_t - c^2 \hat{u}_{xx} = u_t - c^2 u_{xx} = 0$

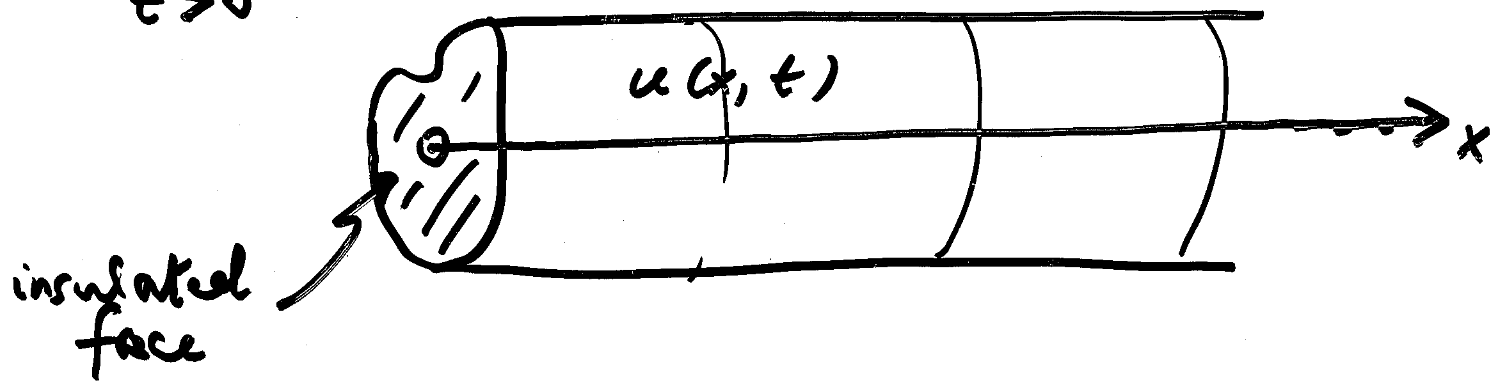
IC $\underset{0<x<\infty}{u(x,0) = F(x)} \longrightarrow \underset{0<x<\infty}{\hat{u}(x,0) = F(x) - u_0 = \hat{F}(x)}$

BC $\underset{t>0}{u(0,t) = u_0} \longrightarrow \underset{t>0}{\hat{u}(0,t) = 0}$

Solve for $\underset{\substack{0<x<\infty \\ t>0}}{\hat{u}(x,t)}$ as above, then

$u(x,t) = \hat{u}(x,t) + u_0$

Suppose instead of BC $\underset{t>0}{u(0,t)=0}$ we have $\underset{t>0}{u_x(0,t)=0}$. (Insulation condition)



So we have: $u_t(x,t) = c^2 u_{xx}(x,t)$, $0<x<\infty$, $t>0$

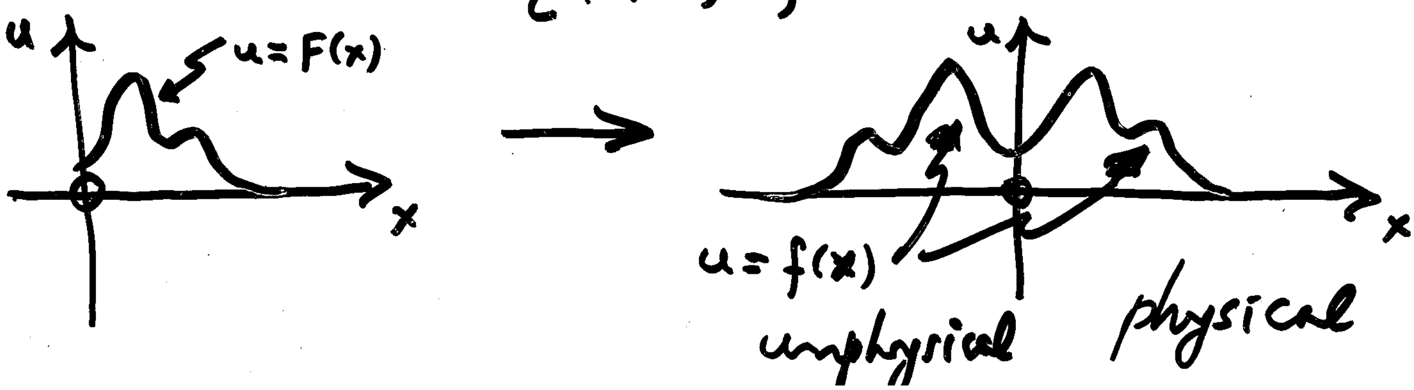
IC: $u(x,0) = F(x)$, $0<x<\infty$

BC: $u_x(0,t) = 0$, $t>0$

This time, trick is to extend to problem on whole x-axis with <u>even</u> initial data.

So: set

$$f(x) = \begin{cases} F(x), & 0<x<\infty \\ F(-x), & -\infty<x<0 \end{cases}$$



Again, solution of extended problem also gives solution of original problem, when restricted to $0<x<\infty$

Solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4c^2t} f(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^{\infty} e^{-(x-y)^2/4c^2t} F(y)\,dy + \int_{-\infty}^{0} e^{-(x-y)^2/4c^2t} F(-y)\,dy \right\}$$

this time get

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^{\infty} \left[e^{-(x-y)^2/4c^2t} + e^{-(x+y)^2/4c^2t} \right] F(y)\,dy \right. \qquad (30.3)}$$

or

$$\boxed{u(x,t) = \int_0^{\infty} \left[G(x-y,t) + G(x+y,t) \right] F(y)\,dy \qquad (30.4)}$$

Let's check that the BC is indeed satisfied:

$$(30.3) \Rightarrow u_x(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^\infty \left[-\frac{(x-y)}{2c^2 t} e^{-(x-y)^2/4c^2 t} - \frac{(x+y)}{2c^2 t} e^{-(x+y)^2/4c^2 t} \right] F(y)\, dy \right.$$

$$\Rightarrow u_x(0,t) = \frac{1}{\sqrt{4\pi c^2 t}} \left\{ \int_0^\infty \left[\frac{y}{2c^2 t} e^{-y^2/4c^2 t} - \frac{y}{2c^2 t} e^{-y^2/4c^2 t} \right] F(y)\, dy \right.$$

$$= 0.$$

<u>Summary</u>:

1) Familiarise yourself with properties of $\mathrm{erf}(z)$, $\mathrm{erf}\left(\frac{x}{\sqrt{4c^2 t}}\right)$

2) Be able to use change of variable to express solutions of 1-D heat equation on whole X-axis in terms of <u>erf</u>

3) Understand <u>Method of Images</u> for BC

a) $u(0,t) = 0$, $t > 0$ b) $u_x(0,t) = 0$, $t > 0$

with 1-D Heat Equation on semiaxis $0 < x < \infty$