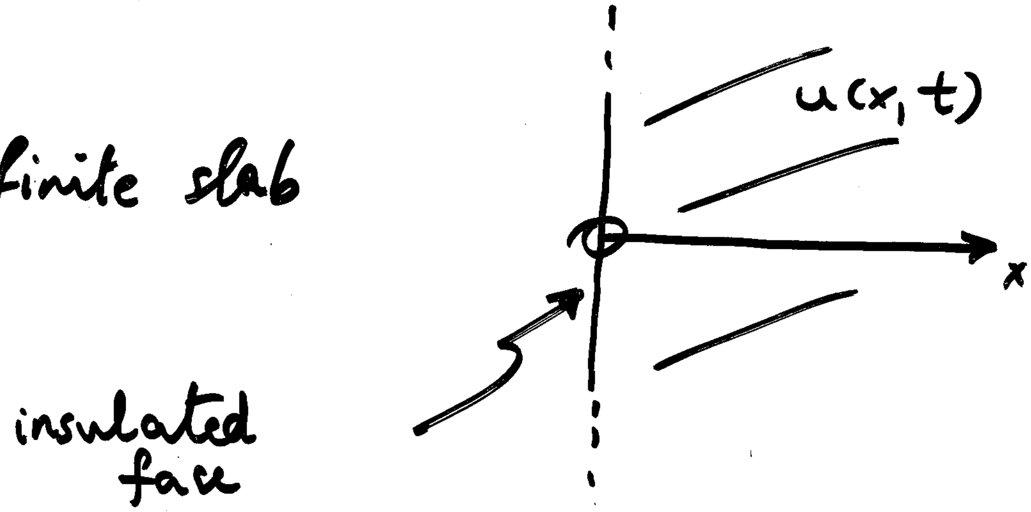
MATH2100 Lec 31 (= MATH2011 Lec 13)

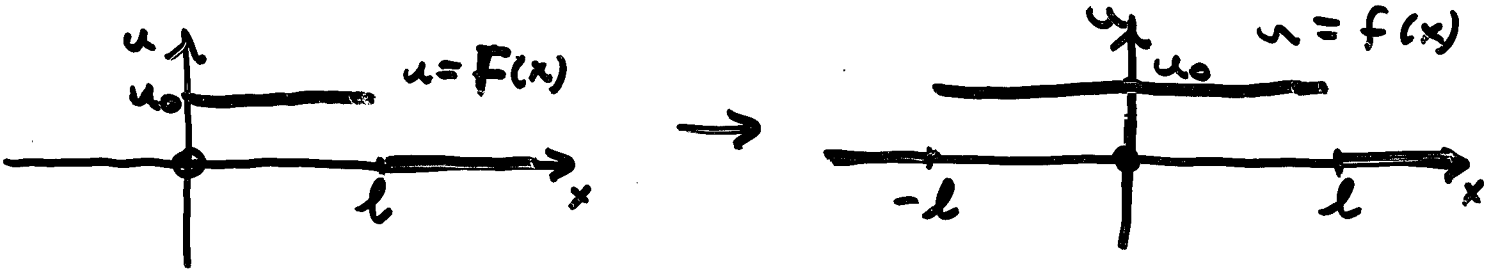
EX: Semi-infinite slab



PDE: $u_t(x,t) = c^2 u_{xx}(x,t) \qquad 0 < x < \infty, \qquad t > 0$

IC: $u(x,0) = F(x) = \begin{cases} u_0, & 0 < x < l \\ 0 & x > l \end{cases}$

BC: $u_x(0,t) = 0, \quad t > 0$



$F(x), \quad 0 < x < \infty \quad \longrightarrow \quad f(x) = \begin{cases} F(x), & 0 < x < \infty \\ F(-x), & -\infty < x < 0 \end{cases}$

(Exercise)

Now have Problem 2 on Tut. sheet 9.

→ solution

$$u(x,t) = \tfrac{1}{2} u_0 \left\{ \operatorname{erf}\left(\frac{x+l}{\sqrt{4c^2t}}\right) - \operatorname{erf}\left(\frac{x-l}{\sqrt{4c^2t}}\right) \right\}$$

— but now for $x > 0, \quad t > 0$.

$l \to a \qquad c^2 \to D \qquad u(x,t) \to c(x,t)$

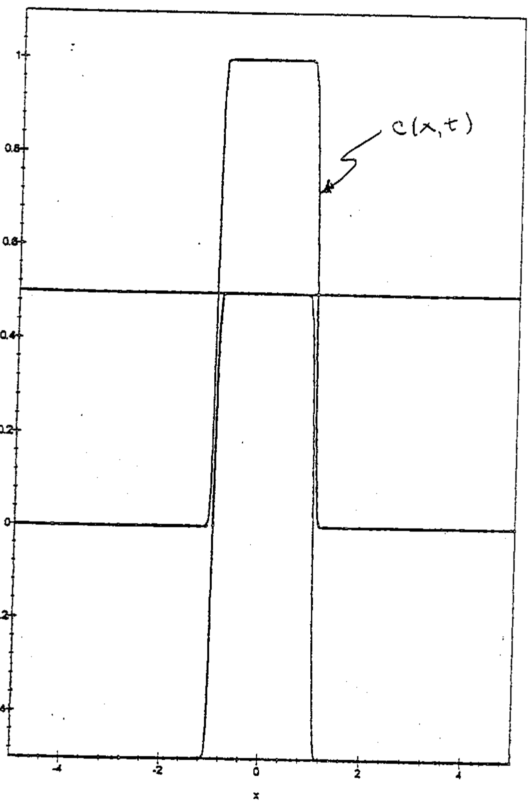
$\qquad\qquad\qquad\qquad\qquad\qquad u_0 \to c_0$

31.2

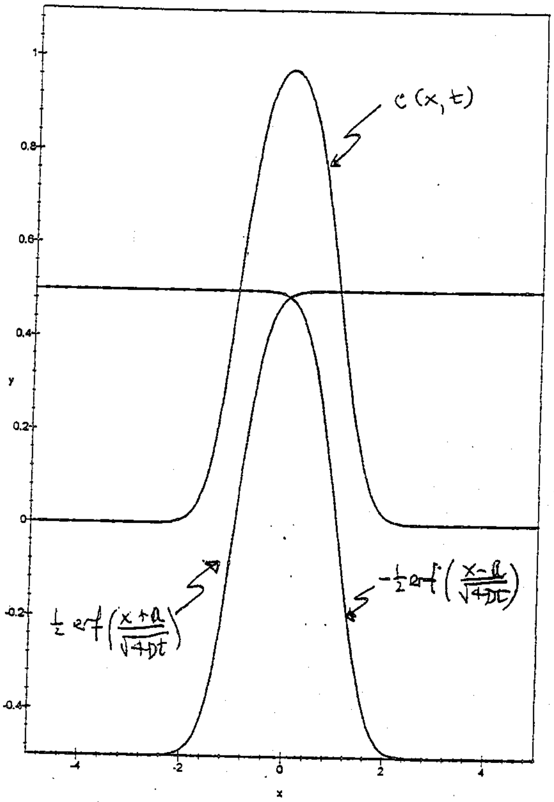
Graphs of $\frac{1}{2}\operatorname{erf}\left(\frac{x+a}{\sqrt{4Dt}}\right)$, $-\frac{1}{2}\operatorname{erf}\left(\frac{x-a}{\sqrt{4Dt}}\right)$ and

$$c(x,t) = \frac{1}{2} c_0 \left[\operatorname{erf}\left(\frac{x+a}{\sqrt{4Dt}}\right) - \operatorname{erf}\left(\frac{x-a}{\sqrt{4Dt}}\right)\right]. \qquad \left\{\begin{matrix} c_0 = 1 \\ D = 1 \\ a = 1 \end{matrix}\right\}$$

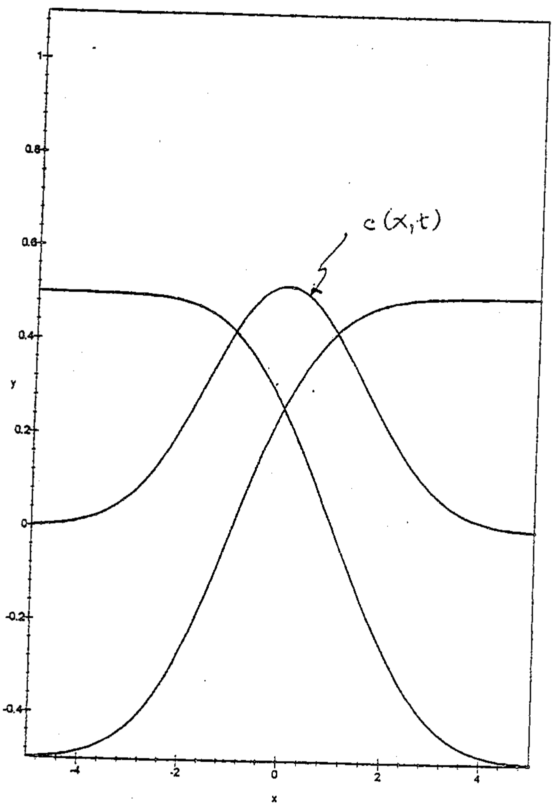
$(t = 0.001)$



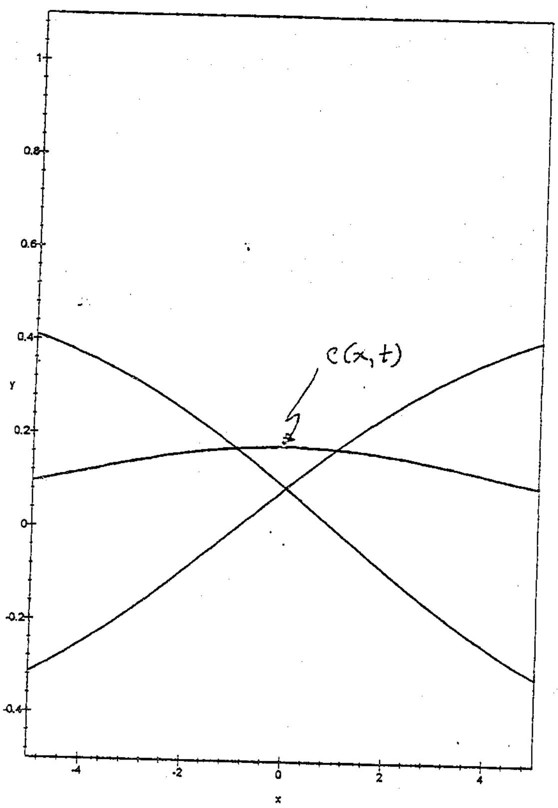
$(t = 0.1)$



$(t = 1)$



$(t = 10)$



Conduction of heat sometimes called '<u>thermal diffusion</u>.' The same PDE governs process of <u>molecular diffusion</u>.

Now $u(x,t) \to c(x,t)$ concentration of diffusate

$c^2 \to D$ coeff. of diffusion

$u_t = c^2 u_{xx} \longrightarrow c_t(x,t) = D\, c_{xx}(x,t)$

1-D Diffusion Equation.

Typical value of $D \sim 10^{-5}\,\text{cm}^2/\text{sec}$

[$\Rightarrow$ over length scales $\sim 5\,\text{cm}$,

typical diffusion times $\sim \frac{25}{10^{-5}}$ sec $\sim$ 30 days;

over length scales $\sim 10^{-4}\,\text{cm}$,

typical diffusion times $\sim \frac{10^{-8}}{10^{-5}}$ sec $\sim 10^{-3}$ sec.

— very important transport process for cells.]

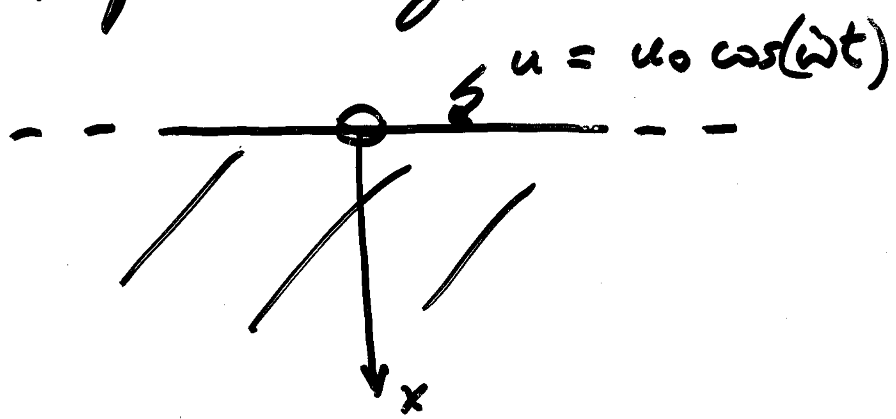
Before leaving heat conduction/diffusion, consider:

Temperature waves in the earth

(The earthworm's Christmas problem)

We consider variation of temperature in soil as a result of periodic variation of temperature at surface (a) daily or (b) yearly.

Model as semi-infinite region with sinusoidal BC:



We have PDE: $u_t(x,t) = c^2 u_{xx}(x,t)$ (31.1)

$0 < x < \infty$

$-\infty < t < \infty$

BC: $u(0,t) = u_0 \cos(\omega t)$ (31.2)

IC: irrelevant – we are interested in steady-state behaviour.

Solution should be periodic in time with angular frequency ω.

We consider the PDE for <u>complex-valued</u> u, with the <u>complex</u> BC $u(0,t) = u_0 e^{i\omega t}$ we look for a complex solution in the form

$$u(x,t) = X(x) e^{i\omega t}$$

The real part will then provide the desired solution.

we have
$$u_t(x,t) = i\omega X(x) e^{i\omega t}$$
$$u_{xx}(x,t) = X''(x) e^{i\omega t}$$

So $u_t = c^2 u_{xx} \Rightarrow i\omega X(x) \cancel{e^{i\omega t}} = c^2 X''(x) \cancel{e^{i\omega t}}$

$$\Rightarrow X''(x) = \frac{i\omega}{c^2} X(x) = \alpha^2 X(x)$$

where
$$\alpha = \pm \left(\frac{i\omega}{c^2}\right)^{\frac{1}{2}} = \pm \sqrt{\frac{\omega}{c^2}} \, \frac{(1+i)}{\sqrt{2}}$$

Then
$$X(x) = A e^{\sqrt{\frac{\omega}{2c^2}}(1+i)x} + B e^{-\sqrt{\frac{\omega}{2c^2}}(1+i)x}$$

We want our solution to be bounded as $x \to \infty$

$\Rightarrow A = 0$

$\Rightarrow X(x) = Be^{-\sqrt{\frac{\omega}{2c^2}}(1+i)x}$

$\Rightarrow u(x,t) = Be^{-\sqrt{\frac{\omega}{2c^2}}(1+i)x} e^{i\omega t}$

BC: $u(0,t) = u_0 e^{i\omega t} \Rightarrow B = u_0$

So $u(x,t) = u_0 e^{-\sqrt{\frac{\omega}{2c^2}}(1+i)x} e^{i\omega t}$

$= u_0 e^{-\sqrt{\frac{\omega}{2c^2}}x} e^{i(\omega t - \sqrt{\frac{\omega}{2c^2}}x)}$

We want real part:-

$$\boxed{u(x,t) = u_0 e^{-\sqrt{\frac{\omega}{2c^2}}x} \cos\left(\omega t - \sqrt{\frac{\omega}{2c^2}}x\right) \quad (31.3)}$$

[Check directly that this satisfies (31.1), (31.2)] — intro. of complex nos. is a device to simplify the calculation.]

(31.3) describes an <u>attenuated</u> temperature wave — amplitude decreases exponentially as x increases.

<u>Note also</u>: 1) temp. at depth x is out of phase with temp. at surface $x=0$

2) the bigger is ω, the greater is the attenuation, and the smaller the penetration of the wave into the soil

So:

daily variations $\to$ smaller penetration

yearly variations $\to$ larger penetration

<u>Some numbers</u>: $c^2 \sim 2\times 10^{-3}$ cm^2/sec for typical soil

For yearly variations: $\omega \sim \frac{2\pi}{(365)(24)(3600)}$ sec^{-1}

At $x \sim 1m = 100$ cm

$$\sqrt{\frac{\omega}{2c^2}}\, x \sim 0.7 \sim \pi/4$$

$$e^{-\sqrt{\frac{\omega}{2c^2}}x} \sim \frac{1}{2}$$

So, at depth of 1 metre, have $\frac{\pi}{4}$ phase lag, and amplitude attenuation by $\frac{1}{2}$

At depth of 4 metres, have π phase lag, and attenuation by $\frac{1}{16}$.

So: is Winter at 4 metres down when is Summer at surface – and amplitude of temp. variation only a fraction of that at surface.

(→ usefulness of a deep cellar)

For daily variations, ω multiplied by factor 365. And $\sqrt{365} \approx 19$.

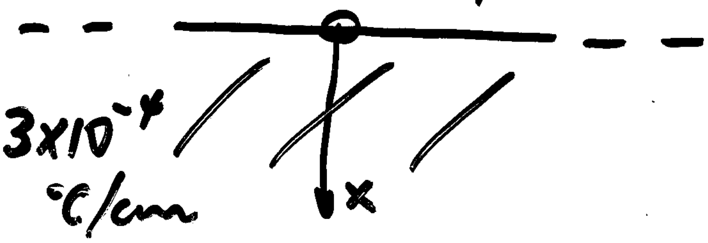
So damping & phase lag which for yearly variations were at x, now are at $\frac{x}{19}$.

So e.g. decrease of amplitude to $\frac{1}{16}$ and 'reversal of time of day' now occur at depth of about $\frac{400}{19} \sim 21$ cm.

– daily variations only in surface skin.

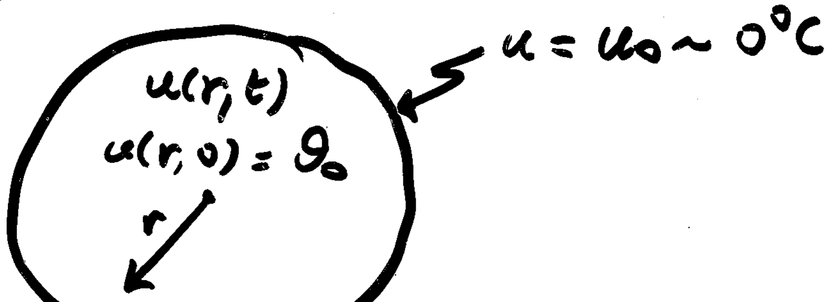
These effects all in topsoil. On a different length scale altogether is effect ~~that~~ temperature steadily increases as bore down into Earth's crust – about 3°C per 100 m.

$u_x(0,t) \sim \frac{3}{100}$ °C/m $= 3\times10^{-4}$ °C/cm



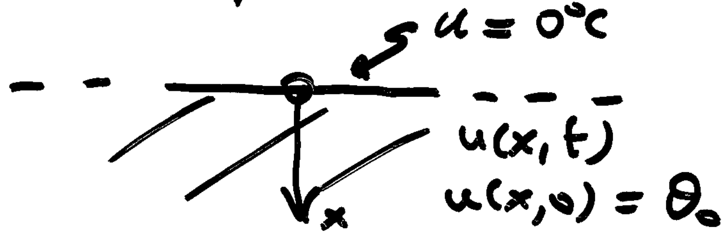
Lord Kelvin used this to estimate age of Earth – very controversial.

He assumed Earth is a hot, chemically inert solid, cooling.



$\theta_0 \sim$ melting temp. for iron ~ 1200°C

Can show effect is near surface – OK to use 'Flat Earth' approximation:–



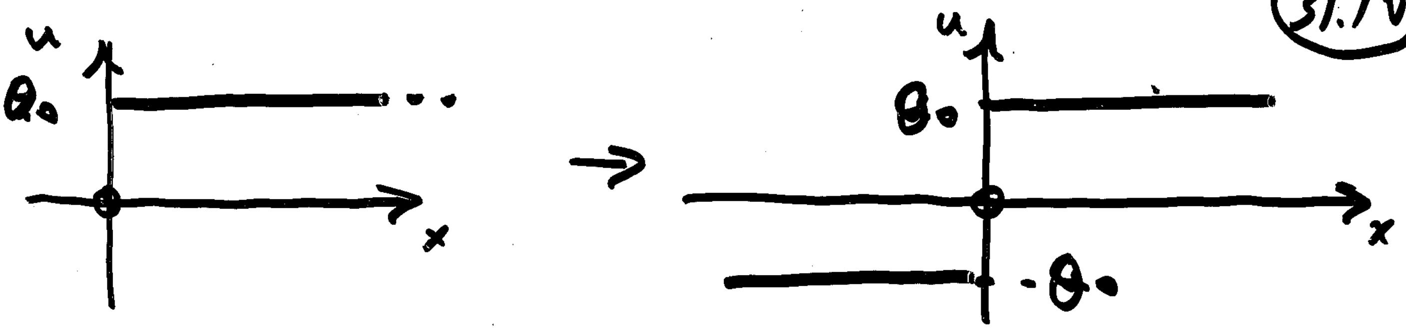


Image problem:
Solved in Lec. 29 (pp. 29.9 – 29.12)

$$u(x,t) = \theta_0 \operatorname{erf}\left(\frac{x}{\sqrt{4c^2t}}\right)$$

$$\Rightarrow \quad u_x(x,t) = \theta_0 \left[\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{4c^2t}}\right)\right] \operatorname{erf}'\left(\frac{x}{\sqrt{4c^2t}}\right)$$

$$= \frac{\theta_0}{\sqrt{4c^2t}} \frac{2}{\sqrt{\pi}} e^{-x^2/4c^2t}$$

$$\Rightarrow \quad u_x(0,t) = \frac{\theta_0}{\sqrt{\pi c^2 t}}$$

$\Rightarrow$ Time Earth has been cooling: $t = \dfrac{(\theta_0)^2}{\pi c^2 [u_x(0,t)]^2}$

Everything on RHS known.
(c^2 for basalt $\sim 6 \times 10^{-3}$ cm^2/sec)

So $\quad t \sim \dfrac{(1200)^2}{(\pi)(6\times10^{-3})(3\times10^{-4})^2}$ secs $\sim 27 \times 10^6$ years

Even allowing for uncertainties in data, Kelvin concluded age of Earth $< 400 \times 10^6$ years.

If we have a source of heat energy present, the 1-D Heat Equation is modified to

$$u_t(x,t) - c^2 u_{xx}(x,t) = \underbrace{q(x,t)}_{\text{`source' term}} \qquad (31.4)$$

– given

$\rho\sigma q(x,t)\delta x$ = amount of heat energy produced in $(x, x+\delta x)$ per unit time.

How can we solve (31.4), say for $0<x<L$, with BCs $u(0,t) = 0 = u(L,t)$, $t>0$, and with IC $u(x,0) = f(x)$, $0<x<L$?

One way to proceed is to expand $u(x,t)$ and $q(x,t)$ in Fourier series, and solve for coefficients in series for u — see example next lecture.

Summary:

1) Know how to use Method of Images for semi-infinite regions with either
a) $u(0,t)=0$ or b) $u_x(0,t)=0$

2) Understand trick of using complex $u(x,t)$ to solve for temperature waves in earth.

3) Follow Kelvin's estimate of Earth's age.