[Exact resting shape of string is \textbf{catenary}:

\[ u^{(65)}(x) = \frac{2}{x} \sinh \left( \frac{\lambda x}{2} \right) \sinh \left( \frac{\lambda (x-\lambda)}{2} \right) \]

\[ = \frac{1}{x} \{ \cosh \left( \frac{\lambda (2x-\lambda)}{2} \right) - \cosh \left( \frac{\lambda \lambda}{2} \right) \} \]

where \( \lambda \) is determined by

\[ \frac{\lambda \lambda}{2} = \sinh \left( \frac{\lambda \lambda}{2} \right) \]

We are interested in situation where \( \lambda L << 1 \), \( \frac{\lambda}{L} \approx 1 \) and so

\[ u^{(65)}(x) \approx \frac{2}{x} \left( \frac{\lambda x}{2} \right) \frac{\lambda (x-\lambda)}{2} = \frac{\lambda x(x-\lambda)}{2} \]

Comparing with p. 32.11, see \( \lambda \approx \frac{g}{2} \) and so for consistency

\[ g \frac{L}{2} << 1 \quad (\iff pgl << T) \quad \text{weight} << \text{tension} \]
We proceed much as in previous example for heat equation:

Suppose \( ICs: \begin{cases} u(x,0) = f(x), & 0 < x < L \\ u_x(x,0) = g(x), & 0 < x < L \end{cases} \)

**BCs:** \( u(0,t) = 0 = u(L,t), \quad t > 0 \)

**PDE:** \( u_{tt} - c^2 u_{xx} = -\varphi, \quad 0 < x < L, \quad t > 0 \)

**Put**

\[
\begin{align*}
\overbrace{u(x,t)}^{\text{displ.}} &= \overbrace{u^{ss}(x)}^{\text{equil.}} + \hat{u}(x,t) \\
\hat{u}(x,t) &= u(x,t) - u^{ss}(x)
\end{align*}
\]
Then:
\[
\hat{u}_{xx}(x,t) - c^2 \hat{u}_{xx}(x,t) = [u_{xx}(x,t) - c^2 u_{xx}(x,t)] - \left[ \frac{u^{(ss)}(x)}{tt_0} - c^2 u^{(ss)}_{xx}(x) \right]
\]
\[
= -g - (-g) = 0
\]

Also:
\[
\begin{align*}
\hat{u}(0,t) &= u(0,t) - u^{(ss)}(0) = 0 - 0 = 0 \\
\hat{u}(L,t) &= u(L,t) - u^{(ss)}(L) = 0 - 0 = 0
\end{align*}
\]

and
\[
\hat{u}(x,0) = u(x,0) - u^{(ss)}(x) = f(x) - u^{(ss)}(x) = f(x), \text{ say.}
\]
\[
\hat{u}_t(x,0) = u_t(x,0) - u^{(ss)}_t(x) = g(x) - 0 = g(x)
\]

So now we have unforced string problem for \( \hat{u}(x,t) \).

**Solution:** (Lec. 27)
\[
\hat{u}(x,t) = \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{n\pi x}{L} \right) + B_n \sin \left( \frac{n\pi x}{L} \right) \right] \sin \left( \frac{n\pi t}{L} \right)
\]

where
\[
A_n = \frac{2}{L} \int_0^L \sin \left( \frac{n\pi x}{2} \right) f(x) \, dx
\]

See (27.14)
and \[ B_n = \left( \frac{k}{\pi a c} \right) \frac{1}{2} \int_0^l \sin \left( \frac{n \pi x}{L} \right) g(x) \, dx \]

Then \[ u(x, t) = \hat{u}(x, t) + u^{ss}(x) \]

\[ u(x, t) = \frac{g}{2c^2} x(x - L) + \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{n \pi c t}{L} \right) \right. \]
\[ + B_n \sin \left( \frac{n \pi c t}{L} \right) \sin \left( \frac{n \pi x}{L} \right) \]

For example, for a plucked string, might have

\[ f(x) = \begin{cases} \frac{2\psi_0 x}{L} & 0 < x < \frac{L}{2} \\ \frac{2\psi_0 (L-x)}{L} & \frac{L}{2} < x < L \end{cases} \]

\[ g(x) = 0 \]

Then \[ B_n = 0, \text{ and} \]

\[ A_n = \frac{1}{2} \int_0^{\frac{L}{2}} \sin \left( \frac{n \pi x}{L} \right) \left[ \frac{2\psi_0 x}{L} - \frac{g}{2c^2} x(x - L) \right] \, dx \]
\[ + \frac{1}{2} \int_{\frac{L}{2}}^L \sin \left( \frac{n \pi x}{L} \right) \left[ \frac{2\psi_0 (L-x)}{L} - \frac{g}{2c^2} x(x - L) \right] \, dx \]
We won't pursue this further. See we can solve the problem, and more complicated ones, in principle.

We note that frequencies of normal modes of vibration are unchanged:

$$\omega_n = \frac{n \pi c}{2L} \quad \text{(or } v_n = \frac{n c}{2L} \text{)}$$

Turn now to

**Steady-state problems in**  
**Heat Conduction in 2-D**

Recall Heat Equation in 3-D:

$$u_t (x, y, z, t) = c^2 \nabla^2 u (x, y, z, t)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{Laplacian operator}$$
For steady temperature distribution:
\[ u(x, y, z) \Rightarrow u_t = 0 \]
and so Heat Equation reduces to
\[ \nabla^2 u(x, y, z) = 0 \]  
Laplace's Equation
or
\[ u_{xx}(x, y, z) + u_{yy}(x, y, z) + u_{zz}(x, y, z) = 0 \]
If no \( y \) or \( z \) - dependence in problem:
reduces to:
\[ u_{xx}(x) = 0 \]
\[ \Rightarrow u(x) = \alpha x + \beta \]

**EX:**
\[ u(0) = u_0 \Rightarrow 0 + \beta = u_0 \Rightarrow u(x) = \alpha x + u_0 \]
\[ u(L) = u_1 \Rightarrow \alpha L + u_0 = u_1 \Rightarrow \alpha = \frac{u_1 - u_0}{L} \]
\[ \Rightarrow u(x) = \left( \frac{u_1 - u_0}{L} \right) x + u_0 \]
2-D situations more interesting

(we won't try 3-D)

\( u(x, y) \) - no \( z \)-dependence

\[ u_{xx}(x, y) + u_{yy}(x, y) = 0 \]

Laplace's Equation in 2-D

Again, we can Fourier's Method in simple cases:

EX: Steady temperature distribution in a long beam of rectangular cross-section:

\[ u = f(x) \]

\( u(x, y) \)

\( u = 0 \) on these three faces
Now no ICs – only BCs. A boundary-value problem (BVP):

PDE: \[ u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad 0 < x < a, \quad 0 < y < b \] (33.2)

BCs:
\[
\begin{align*}
  u(x, 0) &= 0, \quad 0 < x < a \quad -(33.7) \\
  u(x, b) &= f(x), \quad 0 < x < a \quad -(33.6) \\
  u(0, y) &= 0, \quad 0 < y < b \quad -(33.7) \\
  u(a, y) &= 0, \quad 0 < y < b \quad -(33.4) \\
  u(x, 0) &= 0, \quad 0 < x < a \quad -(33.5)
\end{align*}
\]

We look for all possible solutions of the homogeneous equations (33.2), (33.3), (33.4), (33.5) in separated form:

\[ u(x, y) = F(x) G(y) \] (33.7)
First, the homogeneous B.C.s:

\[(33.3) \Rightarrow F(0) G(y) = 0 \Rightarrow F(0) = 0 \] [\text{or } G(y) = 0 \Rightarrow u(x, y) = 0 \text{ trivial}]

\[(33.4) \Rightarrow F(a) G(y) = 0 \Rightarrow F(a) = 0 \] \hspace{1cm} (33.9)

\[(33.5) \Rightarrow F(x) G(0) = 0 \Rightarrow G(0) = 0 \] \hspace{1cm} (33.10)

Next, the PDE:

\[u_{xx} + u_{yy} = 0 \]

\[(33.2) \Rightarrow F''(x) G(y) + F(x) G''(y) = 0 \]

\[\div F(x) G(y) \Rightarrow \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = k \]

So

\[F''(x) = k F(x), \quad G''(y) = -k G(y) \] \hspace{1cm} (33.11)

Next, look at \[(33.11) + (33.8) + (33.9)\]:

Three cases:
Case a): \( k = 0 \)

\[
(33.11) \Rightarrow F''(x) = 0 \Rightarrow F(x) = A x + B
\]

Then \((33.8)\) \[ F(0) = 0 = 0 + B \Rightarrow F(x) = A x \]

Then \((33.9)\) \[ 0 = A a \Rightarrow A = 0 \Rightarrow F(x) = 0 \]

\[ \Rightarrow u(x, y) = 0 \text{ trivial.} \]

Case b): \( k > 0 \) \((k = \mu^2, \text{ say}, \mu > 0)\)

\[
(33.11) \Rightarrow F''(x) = \mu^2 F(x)
\]

\[ \Rightarrow F(x) = A \cosh(\mu x) + B \sinh(\mu x) \]

Then \((33.8)\) \[ 0 = A + 0 \Rightarrow F(x) = B \sinh(\mu x) \]

Then \((33.9)\) \[ 0 = B \sinh(\mu a) \Rightarrow B = 0 \]

\[ \Rightarrow F(x) = 0 \Rightarrow u(x, y) = 0 \text{ trivial} \]

Case c): \( k < 0 \) \((k = -p^2, \text{ say}, p > 0)\)

\[
(33.11) \Rightarrow F''(x) = -p^2 F(x)
\]

\[ \Rightarrow F(x) = A \cos(px) + B \sin(px) \]
Then (33.8) \( \Rightarrow \) \( 0 = A + 0 \Rightarrow F(x) = B \sin (px) \)

Then (33.9) \( \Rightarrow \) \( 0 = B \sin (pa) \Rightarrow B = 0 \Rightarrow F(x) = 0 \)

\( \Rightarrow u(x, y) = 0 \)

**Trivial**

or \( \frac{pa}{a} = n \pi \)

\( p = \frac{n \pi}{a} \quad n = 1, 2, 3, \ldots \)

\( F(x) = B \sin \left( \frac{n \pi x}{a} \right) \)

In this case, \( k = -p^2 = -\left( \frac{n \pi}{a} \right)^2 \)

Then (33.12) \( \Rightarrow \)

\( G''(y) = + \left( \frac{n \pi}{a} \right)^2 G(y) \)

\( \Rightarrow G(y) = C \cosh \left( \frac{n \pi y}{a} \right) + D \sinh \left( \frac{n \pi y}{a} \right) \)

Then (33.10) \( \Rightarrow \)

\( 0 = C + 0 \Rightarrow G(y) = D \sinh \left( \frac{n \pi y}{a} \right) \)

Combining:

\( u(x, y) = B \cdot \sin \left( \frac{n \pi x}{a} \right) \sinh \left( \frac{n \pi y}{a} \right) \)

or

\( u_n (x, y) = B_n \sin \left( \frac{n \pi x}{a} \right) \sinh \left( \frac{n \pi y}{a} \right) \) (33.13)

**Following Fourier, assume** \( n = 1, 2, 3, \ldots \)

**Note:** The text appears to be part of a mathematical derivation or proof, involving trigonometric and hyperbolic functions. The notation and symbols suggest it is from a physics or engineering context, likely involving wave equations or similar problems. The text is handwritten and contains several mathematical expressions and inequalities.
One BC left to fix the $B_n$'s:

\[
\begin{align*}
\text{(33.6)} & \Rightarrow \\
    f(x) = u(x, b) &= \sum_{n=1}^{\infty} B_n \sinh\left(\frac{\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \\
    & \quad 0 < x < a
\end{align*}
\]

Fourier coeffs. for $f(x)$

\[
B_n \sinh\left(\frac{\pi b}{a}\right) = \frac{2}{a \sinh\left(\frac{\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) \, dx
\]

\[
B_n = \frac{2}{a \sinh\left(\frac{\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) \, dx
\]

Summary:

1) Understand extension of Fourier's method to forced string.

2) Understand Fourier's method for 2-D Laplace Equation, rectangular symmetry.

K pp 605-607, 558-560