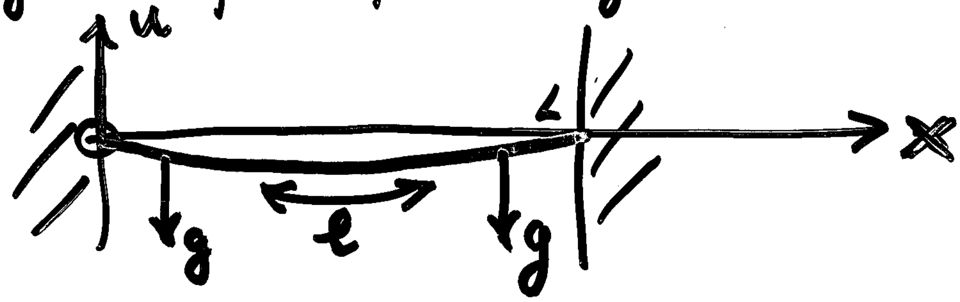
MATH2100 Lec 33 (= MATH2011 Lec. 15)

[Exact resting shape of string is catenary:



$$u^{(ss)}(x) = \frac{2}{\lambda} \sinh\left(\frac{\lambda x}{2}\right) \sinh\left(\frac{\lambda(x-L)}{2}\right)$$

$$\left[= \frac{1}{\lambda}\left\{ \cosh\left(\frac{\lambda(2x-L)}{2}\right) - \cosh\left(\frac{\lambda L}{2}\right)\right\} \right]$$

where λ is determined by

$$\frac{\lambda \ell}{2} = \sinh\left(\frac{\lambda L}{2}\right)$$

We are interested in situation where $\lambda L << 1$, $\frac{\ell}{L} \approx 1$ and so $\boxed{\text{using } \sinh(\theta) \approx \theta \text{ if } |\theta| << 1}$

$$u^{(ss)}(x) \approx \frac{2}{\lambda}\left(\frac{\lambda x}{2}\right) \frac{\lambda(x-L)}{2} = \frac{\lambda x(x-L)}{2}$$

Comparing with p. 32.11, see $\lambda \approx \frac{g}{c^2}$ and so for consistency

$$\frac{gL}{c^2} << 1 \quad (\Leftrightarrow \rho g L << T)$$

weight << tension]

We proceed much as in previous example for heat equation:

Suppose ICs: $\begin{cases} u(x,0) = f(x), & 0 < x < L \\ u_t(x,0) = g(x), & 0 < x < L \end{cases}$

BCs: $u(0,t) = 0 = u(L,t)$, $t > 0$

PDE: $u_{tt} - c^2 u_{xx} = -g$ $\quad 0 < x < L$, $t > 0$

Put

$$\underbrace{u(x,t)}_{\text{displ.}} = \underbrace{u^{(ss)}(x)}_{\text{equilib. displ.}} + \underbrace{\hat{u}(x,t)}_{\text{displ. from equilib.}}$$

$$\Leftrightarrow \quad \hat{u}(x,t) = u(x,t) - u^{(ss)}(x)$$

Then: $\hat{u}_{tt}(x,t) - c^2 \hat{u}_{xx}(x,t)$

$$= \left[u_{tt}(x,t) - c^2 u_{xx}(x,t)\right] - \left[\underbrace{\cancel{u^{(ss)}_{tt}(x)}}_{0} - c^2 u^{(ss)}_{xx}(x)\right]$$

$$= -g - (-g) = 0$$

Also: (New BCs)
$$\begin{cases} \hat{u}(0,t) = u(0,t) - u^{(ss)}(0) = 0 - 0 = 0 \\ \hat{u}(L,t) = u(L,t) - u^{(ss)}(L) = 0 - 0 = 0 \end{cases}$$

and
$$\hat{u}(x,0) = u(x,0) - u^{(ss)}(x) = f(x) - u^{(ss)}(x) = \hat{f}(x), \text{ say.}$$

$$\hat{u}_t(x,0) = u_t(x,0) - u^{(ss)}_t(x) = g(x) - 0 = g(x)$$

So now we have unforced string problem for $\hat{u}(x,t)$.

Solution: (Lec. 27)

$$\hat{u}(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

Where $A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\hat{f}(x)\,dx$ see (27.10)

and

$$B_n = \left(\frac{L}{n\pi c}\right)\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) g(x)\,dx$$

Then $u(x,t) = \hat{u}(x,t) + u^{(ss)}(x)$

$$u(x,t) = \frac{g}{2c^2}x(x-L) + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

For example, for plucked string, might have

$$f(x) = \begin{cases} \frac{2u_0 x}{L}, & 0 < x < \frac{L}{2} \\ \frac{2u_0}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

$$g(x) = 0$$

Then $B_n = 0$, and

$$A_n = \frac{2}{L}\int_0^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{2u_0 x}{L} - \frac{g}{2c^2}x(x-L)\right]dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{2u_0}{L}(L-x) - \frac{g}{2c^2}x(x-L)\right]dx$$

We won't pursue this further. See we can solve the problem, and more complicated ones, in principle.

We note that frequencies of normal modes of vibration are unchanged:

$$\omega_n = \frac{n\pi c}{L} \qquad \left(\text{or} \quad \nu_n = \frac{nc}{2L}\right)$$

Turn now to

Steady-state problems in Heat Conduction in 2-D

Recall Heat Equation in 3-D:

$$u_t(x,y,z,t) = c^2 \nabla^2 u(x,y,z,t)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{Laplacian operator}$$

For steady temperature distribution:

$$u(x,y,z) \Rightarrow u_t = 0$$

and so Heat Equation reduces to

$$\nabla^2 u(x,y,z) = 0 \qquad \text{Laplace's Equation}$$

or

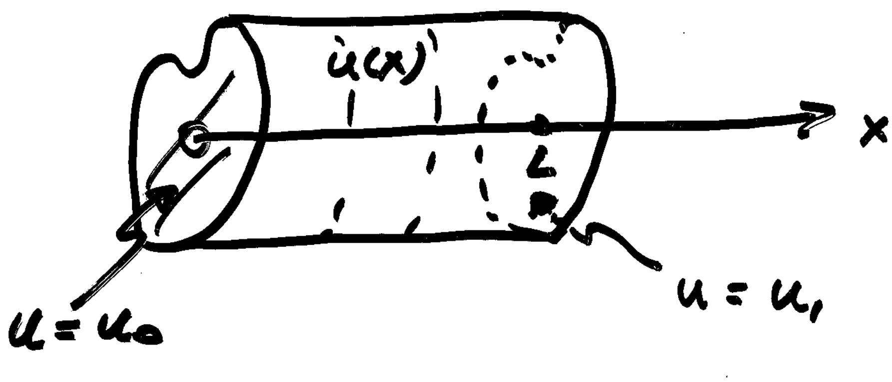
$$u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z) = 0$$

If no y or z-dependence in problem: reduces to:

$$u_{xx}(x) = 0$$

$$\Rightarrow \quad u(x) = \alpha x + \beta$$

__EX:__



$$u(0) = u_0 \Rightarrow 0 + \beta = u_0 \Rightarrow u(x) = \alpha x + u_0$$

$$u(L) = u_1 \Rightarrow \alpha L + u_0 = u_1 \Rightarrow \alpha = \frac{u_1 - u_0}{L}$$

So $\quad u(x) = \left(\frac{u_1 - u_0}{L}\right)x + u_0$

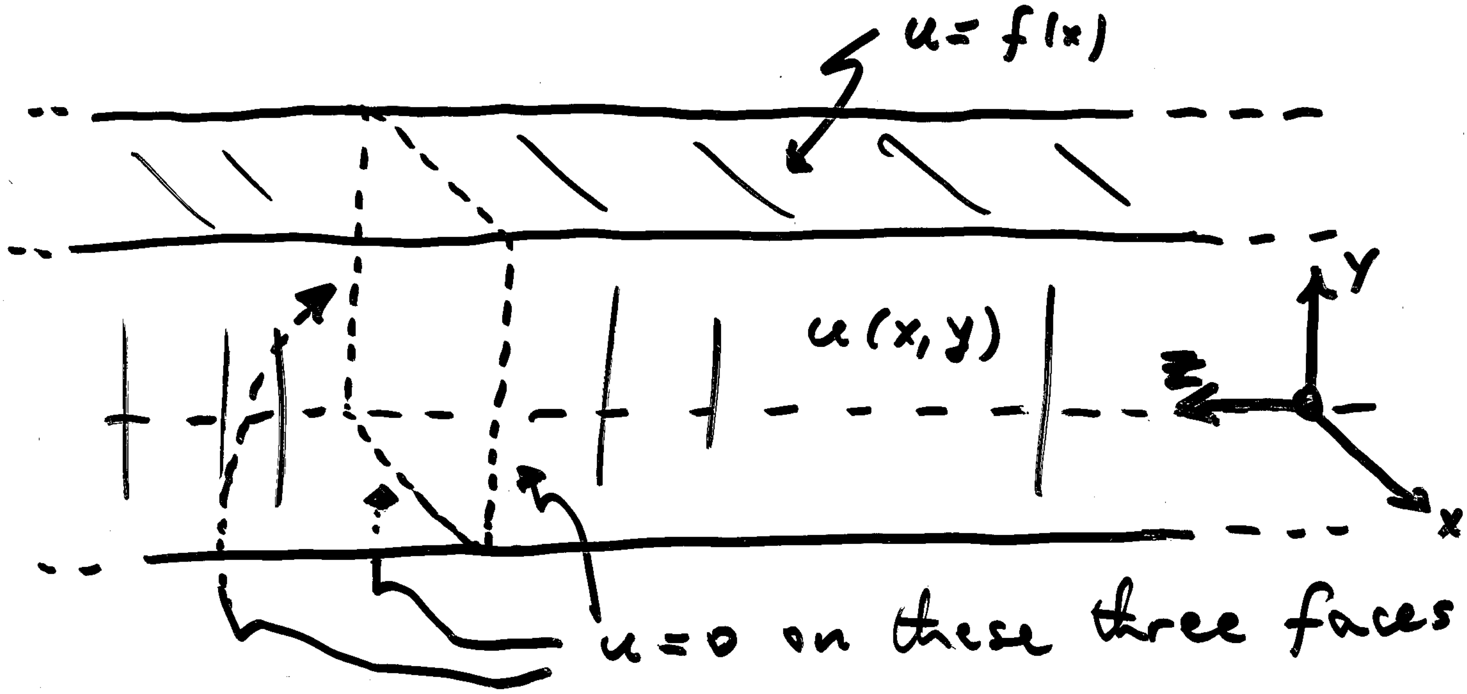
2-D situations more interesting
(we won't try 3-D)

$u(x, y)$ – no z- dependence

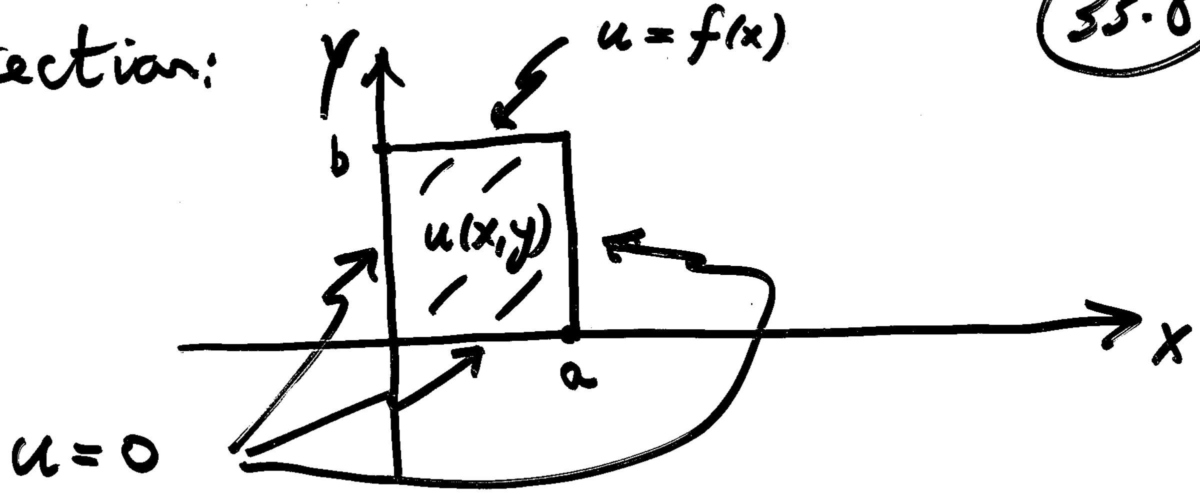
$$\Rightarrow \boxed{u_{xx}(x, y) + u_{yy}(x, y) = 0 \quad \text{Laplace's Equation in 2-D}} \qquad (33.1)$$

Again, we can Fourier's Method
in simple cases:

<u>EX</u>: Steady temperature distribution
in a long beam of rectangular
cross- section:



Cross-section:



Now no ICs – only BCs. A boundary-value problem (BVP):

PDE:
$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad \begin{array}{l} 0<x<a \\ 0<y<b \end{array} \qquad (33.2)$$

BCs:
$$\begin{cases} u(0,y) = 0, & 0<y<b \quad \text{—}(33.3) \\ u(a,y) = 0, & 0<y<b \quad \text{—}(33.4) \\ u(x,0) = 0, & 0<x<a \quad \text{—}(33.5) \\ u(x,b) = f(x), & 0<x<a \quad \text{—}(33.6) \end{cases}$$

We look for all possible solutions of the <u>homogeneous</u> equations (33.2), (33.3), (33.4), (33.5) in separated form:

$$\boxed{u(x,y) = F(x)\,G(y) \qquad (33.7)}$$

First, the homogeneous BCs:

$$(33.3) \Rightarrow \underbrace{u(0,y)}= F(0)G(y) = 0 \Rightarrow \underline{F(0) = 0} \qquad (33.8)$$

$$\left[\text{or } G(y) = 0 \Rightarrow u(x,y) = 0 \text{ trivial}\right]$$

$$(33.4) \Rightarrow \underbrace{u(a,y)} = F(a)G(y) = 0 \Rightarrow \underline{F(a) = 0} \qquad (33.9)$$

$$(33.5) \Rightarrow \underbrace{u(x,0)} = F(x)G(0) = 0 \Rightarrow \underline{G(0) = 0} \qquad (33.10)$$

Next, the PDE:

$$(33.2) \Rightarrow \overset{u_{xx}}{F''(x)G(y)} + \overset{u_{yy}}{F(x)G''(y)} = 0$$

$$\div F(x)G(y) \Rightarrow \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = k$$

So

$$\underline{F''(x) = kF(x)} \quad (33.11), \qquad \underline{G''(y) = -kG(y)} \quad (33.12)$$

Next, look at (33.11) + (33.8) + (33.9):

Three cases:

Case a): $k = 0$

$(33.11) \Rightarrow F''(x) = 0 \Rightarrow F(x) = Ax + B$

Then $(33.8) \Rightarrow \underset{F(0)=}{0} = 0 + B \Rightarrow F(x) = Ax$

Then $(33.9) \Rightarrow \underset{F(a)=}{0} = Aa \Rightarrow A = 0 \Rightarrow F(x) = 0$

$\Rightarrow u(x,y) = 0$ trivial.

Case b): $k > 0$ ($k = \mu^2$, say, $\mu > 0$)

$(33.11) \Rightarrow F''(x) = \mu^2 F(x)$

$\Rightarrow F(x) = A\cosh(\mu x) + B\sinh(\mu x)$

Then $(33.8) \Rightarrow 0 = A + 0 \Rightarrow F(x) = B\sinh(\mu x)$

Then $(33.9) \Rightarrow 0 = B\sinh(\mu a) \Rightarrow B = 0$

$\Rightarrow F(x) = 0 \Rightarrow u(x,y) \equiv 0$ trivial

Case c): $k < 0$ ($k = -p^2$, say, $p > 0$)

$(33.11) \Rightarrow F''(x) = -p^2 F(x)$

$\Rightarrow F(x) = A\cos(px) + B\sin(px)$

Then (33.8) $\Rightarrow$ $0 = A + 0 \Rightarrow F(x) = B\sin(px)$

Then (33.9) $\Rightarrow$ $0 = B\sin(pa) \Rightarrow B = 0 \Rightarrow F(x) = 0$
$\Rightarrow u(x,y) = 0$
trivial

OR $pa = n\pi$
$p = \frac{n\pi}{a}$ $\quad n = 1, 2, 3 \ldots$

$$F(x) = B\sin\left(\frac{n\pi x}{a}\right)$$

In this case, $k = -p^2 = -\left(\frac{n\pi}{a}\right)^2$

Then (33.12) $\Rightarrow$ $G''(y) = +\left(\frac{n\pi}{a}\right)^2 G(y)$

$$\Rightarrow G(y) = C\cosh\left(\frac{n\pi y}{a}\right) + D\sinh\left(\frac{n\pi y}{a}\right)$$

Then (33.10) $\Rightarrow$ $\underset{G(0)}{0} = C + 0 \Rightarrow G(y) = D\sinh\left(\frac{n\pi y}{a}\right)$

Combining:
$$u(x,y) = B.D.\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

or

$$\boxed{u_n(x,y) = B_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \quad (33.13)}$$

Following Fourier, assume $\quad n = 1, 2, 3 \ldots$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)} \quad (33.14)$$

One BC left to fix the B_n's:

(33.6) $\Rightarrow$

$$f(x) = u(x,b) = \sum_{n=1}^{\infty} \underbrace{B_n \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right)$$

$0 < x < a$

Fourier coeffs. for $f(x)$

So $$B_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$\boxed{B_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx} \quad (33.15)$$

<u>Summary</u>:

1) Understand <u>extension</u> of Fourier's method to forced string.
2) Understand Fourier's method for 2-D Laplace Equation, rectangular symmetry.

K pp ~~605 – 607~~. 558 – 560