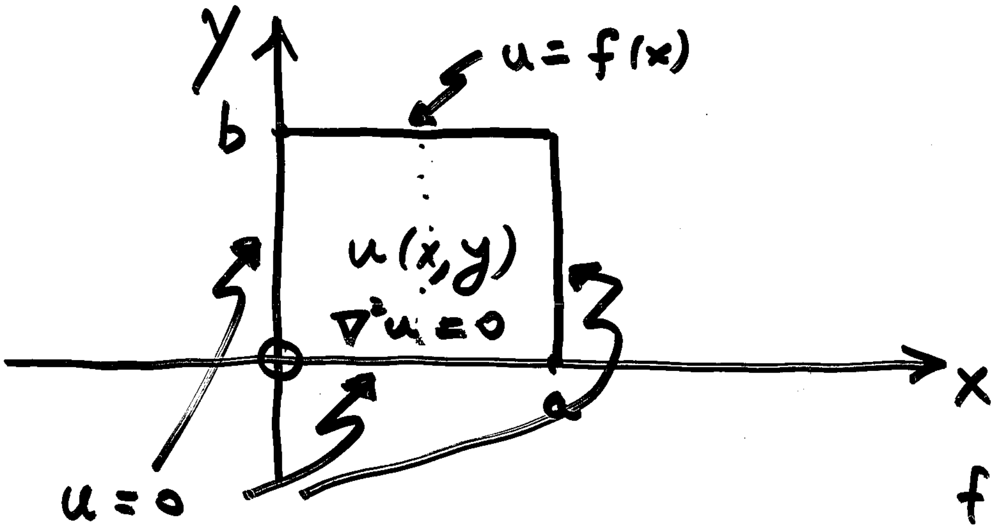
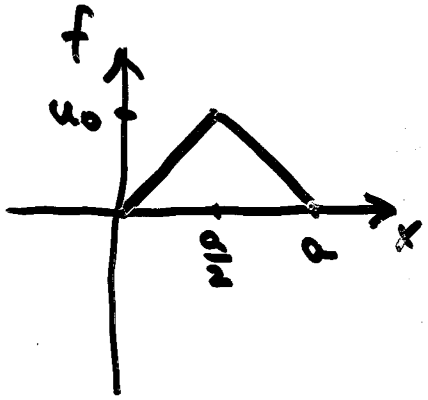
MATH2100 Lec 34 (= MATH2011 Lec. 16)

EX:



$$f(x) = \begin{cases} \frac{2u_0}{a}x, & 0 < x < \frac{a}{2} \\ \frac{2u_0}{a}(a-x), & \frac{a}{2} < x < a \end{cases}$$



Might guess, by symmetry,

$$u(x,y) = \frac{y}{b} f(x) = \begin{cases} \frac{2u_0}{ab} xy, & 0 < x < \frac{a}{2},\ 0 < y < b \\ \frac{2u_0}{ab}(a-x)y, & \frac{a}{2} < x < a,\ 0 < y < b \end{cases} \tag{34.1}$$

Easy to see this satisfies all BCs, and also the PDE, <u>except</u> on the line $x = \frac{a}{2}$

It is <u>not</u> the solution.

Let's see what happens. We have

$$u(x,y)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \qquad (34.2)$$

Where

$$B_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\left\{\int_0^{\frac{a}{2}} \frac{2u_0}{a}\, x \sin\left(\frac{n\pi x}{a}\right)dx + \int_{\frac{a}{2}}^{a} \frac{2u_0}{a}(a-x)\sin\left(\frac{n\pi x}{a}\right)dx\right\}$$

$$= \frac{4u_0}{a^2\sinh\left(\frac{n\pi b}{a}\right)}\left\{-\left[x\left(\frac{a}{n\pi}\right)\cos\left(\frac{n\pi x}{a}\right)\right]_0^{\frac{a}{2}} + \left(\frac{a}{n\pi}\right)^2\left[\sin\left(\frac{n\pi x}{a}\right)\right]_0^{\frac{a}{2}} - \left[(a-x)\left(\frac{a}{n\pi}\right)\cos\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a} - \left(\frac{a}{n\pi}\right)^2\left[\sin\left(\frac{n\pi x}{a}\right)\right]_{\frac{a}{2}}^{a}\right\}$$

Using

$$\int x\sin(px)\,dx = -\frac{x}{p}\cos(px) + \frac{1}{p^2}\sin(px)$$

$$= \frac{4u_0}{a^2\sinh\left(\frac{n\pi b}{a}\right)}\left\{\cancel{-\left(\frac{a}{2}\right)\left(\frac{a}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right)} + \left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) \cancel{+\left(\frac{a}{2}\right)\left(\frac{a}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right)} + \left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right\}$$

$$= \frac{8u_0}{(n\pi)^2\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi}{2}\right)$$

Now

$\sin\left(\frac{n\pi}{2}\right)$:	1	0	−1	0	1	0	−1	...
n :	1	2	3	4	5	6	7	...

$$\Rightarrow \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n = 2m \text{ (even)} \\ (-1)^m & n = 2m+1 \text{ (odd)} \end{cases} \quad m = 0, 1, 2 \ldots$$

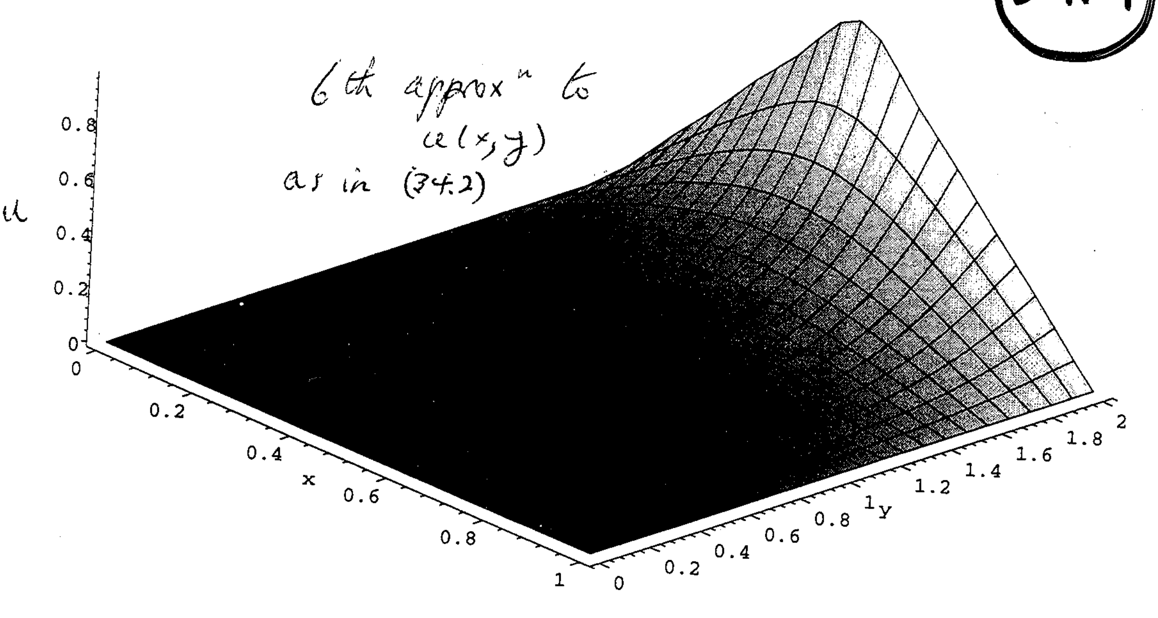
So:

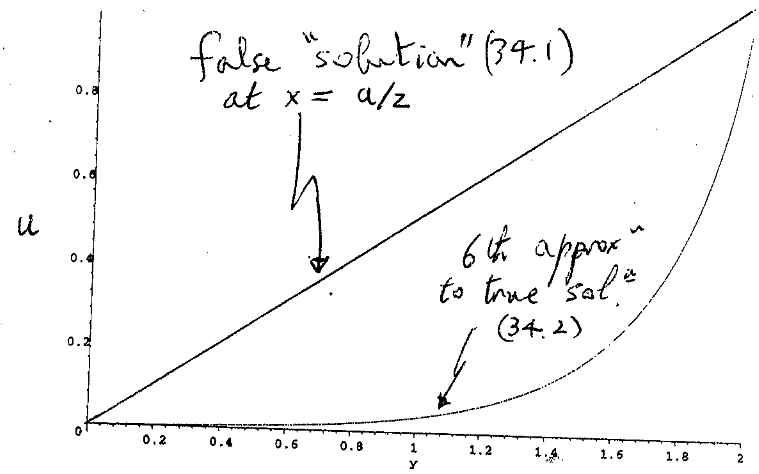
$$\boxed{u(x,y) = \frac{8u_0}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m \sin\left[\frac{(2m+1)\pi x}{a}\right] \sinh\left[\frac{(2m+1)\pi y}{a}\right]}{(2m+1)^2 \sinh\left(\frac{(2m+1)\pi b}{a}\right)}}$$

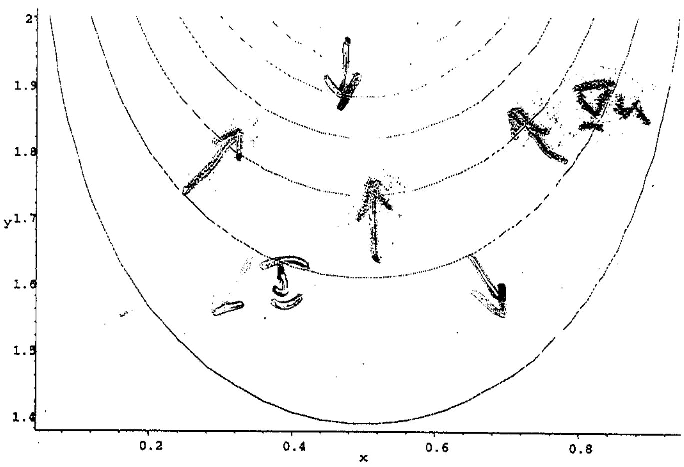
$$\boxed{= \frac{8u_0}{\pi^2}\left[\frac{\sin\left(\frac{\pi x}{a}\right)\sinh\left(\frac{\pi y}{a}\right)}{\sinh\left(\frac{\pi b}{a}\right)} - \frac{\sin\left(\frac{3\pi x}{a}\right)\sinh\left(\frac{3\pi y}{a}\right)}{9\sinh\left(\frac{3\pi b}{a}\right)} + \ldots\right]} \quad (34.2)$$

The next page shows a MAPLE plot3d of the sum of the first 6 terms. See the solution is <u>very</u> <u>smooth</u> inside the rectangle $0<x<a$, $0<y<b$, in contrast to (34.1)

This is a general feature of Laplace's equation in 2-D:





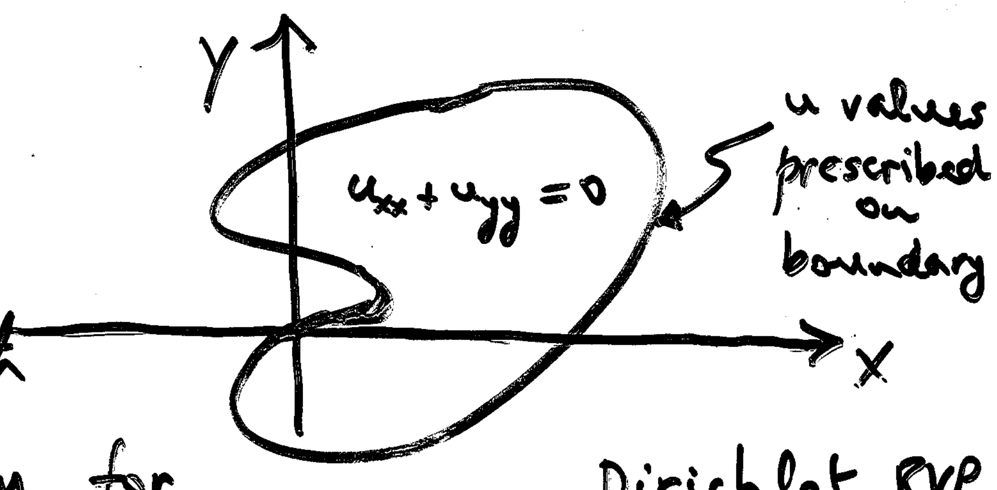


Contour (level) lines of u (isotherms)

[Here $u_0 = 1$, $a = 1$, $b = 2$.]

<u>In general:</u>

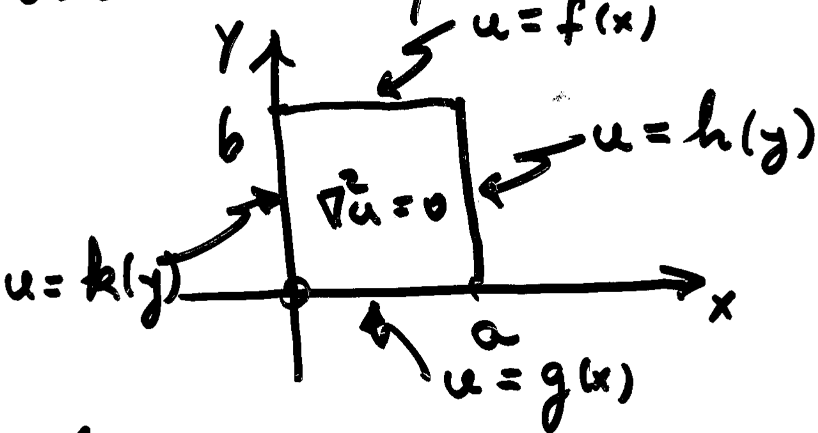
Can show $u(x,y)$ is (infinitely) smooth inside the boundary, for fairly general u on boundary.

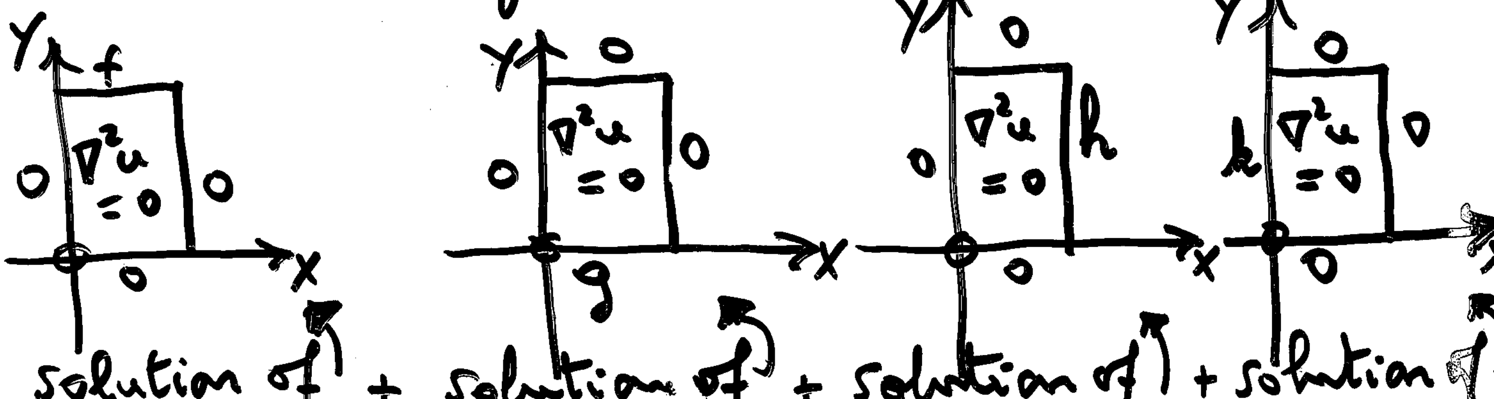


<u>Dirichlet BVP</u>

[Another interesting property of the Dirichlet BVP for Laplace's Equation is that $u(x,y)$ in interior does not take any values {smaller / bigger} than the {smallest / largest} value u takes on the boundary.]

<u>Returning to our example</u>: Now see solution of





= solution of + solution of + solution of + solution of

$$u(x,y) = u^{(1)}(x,y) + u^{(2)}(x,y) + u^{(3)}(x,y) + u^{(4)}(x,y)$$

[Just think about BCs to see this]

Recall def^{n} of <u>level surfaces</u> of scalar field $\varphi(x,y,z)$: surfaces on which $\varphi = \text{const.}$

In 2-D we have <u>level lines</u> (contour lines) of $u(x,y)$: curves on which $u = \text{const.}$

As in 3-D, $\underset{\sim}{\nabla}\varphi$ at any point is directed $\perp^{r}$ to level surface through that point, in direction of <u>max. increasing</u> φ,

So in 2-D:

$$\underset{\sim}{\nabla} u(x,y) = \frac{\partial u(x,y)}{\partial x}\underset{\sim}{i} + \frac{\partial u(x,y)}{\partial y}\underset{\sim}{j}$$

at any point is directed $\perp^{r}$ to contour line through that point, in direction of <u>max increasing</u> u. The heat flux vector $\underset{\sim}{J} = -\kappa \underset{\sim}{\nabla} u$ is in the <u>opposite</u> direction.

A function (scalar field) $u(x,y)$ satisfying Laplace's Equation

$$\nabla^2 u(x,y) = 0$$

i.e.

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

throughout some region of the XY-plane is called <u>harmonic</u> on that region.

Harmonic functions have a very interesting and important property: <u>they always come in pairs:</u>

Given $u(x,y)$, harmonic on some region, can always find $v(x,y)$, also harmonic on <u>same</u> region, such that

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \quad \underline{\text{and}} \quad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$$

i.e.

$$\boxed{\begin{array}{c} u_x(x,y) = v_y(x,y) \quad \underline{\text{and}} \quad u_y(x,y) = -v_x(x,y) \quad (34.3) \\ \text{Cauchy-Riemann Equations} \end{array}}$$

EX: $u(x,y) = x^2 - y^2$

$u_x(x,y) = 2x \quad , \quad u_{xx}(x,y) = 2$

$u_y(x,y) = -2y \quad , \quad u_{yy}(x,y) = -2$

$\Rightarrow \quad u_{xx}(x,y) + u_{yy}(x,y) = 0$: u is harmonic on whole XY-plane

Find v?

C-R

$v_y = u_x = 2x \qquad v_x = -u_y = 2y$

$\Downarrow$

$v(x,y) = 2xy + F(x)$

$\Downarrow$

$v_x(x,y) = 2y + F'(x)$

$F'(x) = 0$

$\Downarrow$

$F(x) = c$

$\Downarrow$

$v(x,y) = 2xy + c$

check
$v_{xx}(x,y) + v_{yy}(x,y) = 0$

$u(x,y)$ and $v(x,y)$ are called <u>conjugate harmonic functions.</u>

EX: $u_n(x,y) = \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$

$\Rightarrow$

$u_{nx}(x,y) = \left(\frac{n\pi}{a}\right) \cos\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$

$u_{ny}(x,y) = \left(\frac{n\pi}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \cosh\left(\frac{n\pi y}{a}\right)$

$$\Rightarrow u_{nxx}(x,y) = -\left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

$$u_{nyy}(x,y) = \left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

So $u_{nxx}(x,y) + u_{nyy}(x,y) = 0$: $u_n(x,y)$ is harmonic — everywhere, not just inside rectangle $0<x<a$; $0<y<b$.

Well, we already knew that.

But now look for $v_n(x,y)$:

C-R
$$v_{ny}(x,y) = u_{nx}(x,y) = \left(\frac{n\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

$$\Rightarrow v_n(x,y) = \cos\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) + F(x)$$

$$\Rightarrow v_{nx}(x,y) = -\left(\frac{n\pi}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) + F'(x)$$

But

C-R
$$v_{nx}(x,y) = -u_{ny}(x,y) = -\left(\frac{n\pi}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)$$

$\Rightarrow F'(x) = 0 \Rightarrow F(x) = C$

$$\Rightarrow v_n(x,y) = \cos\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) + C_n$$

— the harmonic function <u>conjugate</u> to $u_n(x,y)$

Now see the harmonic function conjugate to our solution (34.2) is (up to addition of an arbitrary constant)

$$v(x,y) = \frac{8u_0}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m \cos\left[\frac{(2m+1)\pi x}{a}\right] \cosh\left[\frac{(2m+1)\pi y}{a}\right]}{(2m+1)^2 \sinh\left[\frac{(2m+1)\pi b}{a}\right]}$$

If $u(x,y)$ is the temperature in the rectangle, what is the physical significance of $v(x,y)$?

Consider $\underset{\sim}{\nabla} u = u_x(x,y)\,\underset{\sim}{i} + u_y(x,y)\,\underset{\sim}{j}$

$$\underset{\sim}{\nabla} v = v_x(x,y)\,\underset{\sim}{i} + v_y(x,y)\,\underset{\sim}{j}$$

C-R $\Rightarrow$ $\underset{\sim}{\nabla} v = -u_y(x,y)\,\underset{\sim}{i} + u_x(x,y)\,\underset{\sim}{j}$

Then

$$\underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} v = -u_x(x,y)\,u_y(x,y) + u_y(x,y)\,u_x(x,y)$$

$$= 0$$

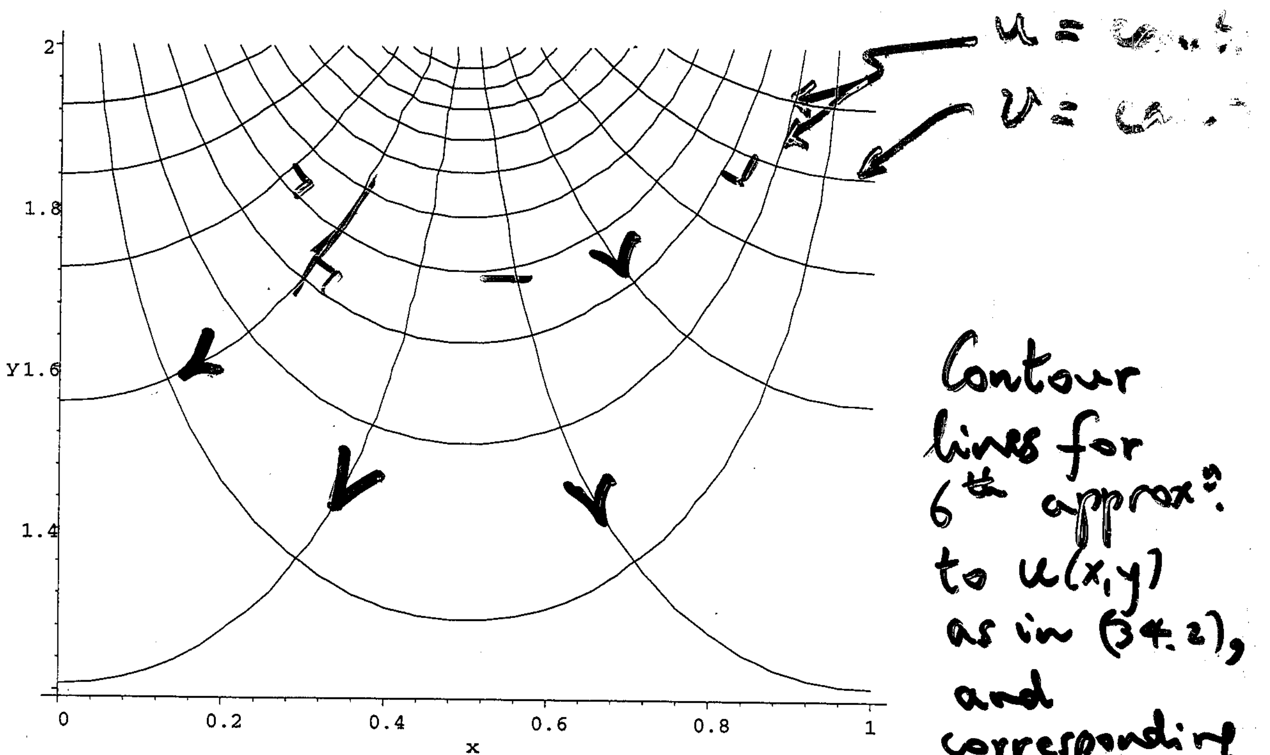
This holds at each point (x,y)

This says that at every point, $\underset{\sim}{\nabla} u$ and $\underset{\sim}{\nabla} v$ are directed $\perp$r to each other.

But $\underset{\sim}{\nabla} u$ is everywhere $\perp$r to contours of u and $\underset{\sim}{\nabla} v$ is everywhere $\perp$r to contours of v

It follows that the contour lines of u are $\perp$r to the contour lines of v, wherever they cross.

Also follows that the contour lines of v are the lines along which heat flows :-



Contour lines for 6^{th} approx.^{n} to $u(x,y)$ as in (34.2), and corresponding $v(x,y)$

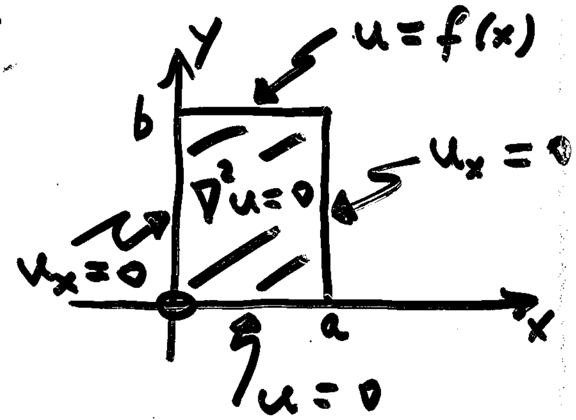
$u(x,y)$: temperature
$v(x,y)$: heat stream function.

Summary:

1) Be able to use Fourier's Method for steady temperatures in beam with rectangular cross-section.

2) Consider also case with:

(Assignment 10).



3) Understand idea of pair of conjugate harmonic functions (u, v) satisfying Cauchy-Riemann Equations
4) Idea of heat stream function.

R.V. Churchill

K: pp ~~605-609, 808, 673~~
558-562, 622, §18.3