A special solution of Laplace's equation in 2-D:

Point source/sink in 2-D

\[ u(x, y) = A \ln \left( x^2 + y^2 \right) \quad (x, y) \neq (0, 0) \quad (35.1) \]

\[
\frac{\partial u(x, y)}{\partial x} = \frac{2A}{(x^2 + y^2)} \cdot 2x, \quad \frac{\partial u(x, y)}{\partial y} = \frac{A \cdot 2y}{(x^2 + y^2)}
\]

\[
\frac{\partial^2 u(x, y)}{\partial x^2} = \frac{2A}{x^2 + y^2} + \frac{2Ax \cdot (-1) \cdot 2x}{(x^2 + y^2)^2}
\]

\[= \frac{2A(x^2 + y^2) - 4Ax^2}{(x^2 + y^2)^2} \]

\[= \frac{2A(y^2 - x^2)}{(x^2 + y^2)^2} \]
Similarly \((x \leftrightarrow y)\)
\[
\frac{\partial^2 u}{\partial y^2} = \frac{2A(x^2-y^2)}{(x^2+y^2)^2}
\]
and so
\[
\nabla^2 u = u_{xx} + u_{yy} = 0
\]
- everywhere except \(x = y = 0\), where \(u\) and its derivatives are not defined.

Level lines of \(u\): lines where
\[
x^2 + y^2 = \text{const.} \quad \Rightarrow \text{circles, centre 0.}
\]
Also
\[
\nabla u = \frac{\partial u}{\partial x} \hat{i} + \frac{\partial u}{\partial y} \hat{j} = \frac{2A}{(x^2+y^2)} (x \hat{i} + y \hat{j})
\]
\[
= \frac{2A}{r^2} \hat{r}
\]
where
\[
r = x \hat{i} + y \hat{j} \quad \text{position vector of point} \ (x, y)\]
So:

\[ |z| = r = \sqrt{x^2 + y^2} \]

Then:

\[ u = Ahn(\alpha^2) \]

Level (contour) lines of \( u \)

\[ \nabla u \]

\( A > 0 \) sink

\( \nabla u \)

\( A < 0 \) source

\[ \nabla u \]

Heat flow

\[ \nabla u \]

\[ J = -k \nabla u \]
Consider heat flowing over circle, radius \( a \), per unit time (per unit length in \( z \)-direction).

(Think of circular cylinder, radius \( a \), axis along \( z \)-axis, \( r \) to page.)

\[
\mathbf{J} \cdot \hat{n} \, ds = -\kappa \cdot \frac{2A}{a^2} \, \hat{r} \cdot \hat{r} \, ds
\]

\[
= -\kappa \cdot \frac{2A}{a^2} \, a \, ds \quad [\hat{r} \cdot \hat{r} = 1]
\]

\[
= -\frac{2\kappa A}{a} \, ds
\]

For whole circle, flux is

\[
-\frac{2\kappa A}{a} \cdot 2\pi a = -4\pi \kappa A
\]

[Note that it doesn't depend on radius of circle. So this is amount of heat flowing out of source, per unit time.]
Choosing $A = -\frac{1}{4\pi k}$, we have

\[
\begin{align*}
    u(x, y) &= -\frac{1}{4\pi k} \ln \left( x^2 + y^2 \right) \\
           &= -\frac{1}{4\pi k} \ln \left( r^2 \right) \\
           &= -\frac{1}{2\pi k} \ln(r) \\
\end{align*}
\]

Temp. distbn. due to point source of heat of unit strength at 0.

Similarly, $u = +\frac{1}{2\pi k} \ln(r)$ is temp. distbn. due to point sink of unit strength at 0.

More generally,

\[
    u(x, y) = \frac{1}{4\pi k} \ln \left[ (x-x_0)^2 + (y-y_0)^2 \right]
\]

is temp. distbn. due to point source of heat of unit strength at $(x_0, y_0)$.

Check: $\nabla^2 u = 0$

$(x, y) \neq (x_0, y_0)$
What about the corresponding heat stream function?

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x} \quad (C-R) \]

\[ \Rightarrow \frac{\partial u}{\partial y} = \frac{2Ax}{x^2+y^2} \quad (1) \quad \frac{\partial u}{\partial x} = -\frac{2Ay}{x^2+y^2} \quad (2) \]

Recall: If \( \rho = \tan \sigma \) \( \leftrightarrow \sigma = \arctan \rho \)

\[ \frac{d\rho}{d\sigma} = 1 + \tan^2 \sigma \quad \leftrightarrow \quad \frac{d\sigma}{d\rho} = \frac{1}{1 + \rho^2} \]

Now \( (1) \Rightarrow \frac{\partial u}{\partial y} = \frac{2A}{x} \frac{1}{1 + (\frac{y}{x})^2} \Rightarrow u(x,y) = 2A \arctan \left( \frac{y}{x} \right) + F(x) \]

\[ \frac{\partial u}{\partial x} = 2A \left( -\frac{y}{x} \right) \frac{1}{1 + (\frac{y}{x})^2} + F'(x) \]

\[ = -\frac{2Ay}{x^2+y^2} + F'(x) \]

Then \( (2) \Rightarrow F'(x) = 0 \Rightarrow F(x) = C \)

So

\[ u(x,y) = 2A \arctan \left( \frac{y}{x} \right) + C \]
Easier to see meaning in polar coordinates:

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

\[
0 < r < \infty \\
0 \leq \theta < 2\pi
\]

Inverse transformation:

\[
\begin{align*}
r &= \sqrt{x^2 + y^2} \\
\tan(\theta) &= \frac{y}{x} \Rightarrow \theta &= \arctan \left( \frac{y}{x} \right)
\end{align*}
\]

So, in polar variables we have

\[
\begin{align*}
u &= A \ln(r^2) = 2A \ln(r) \\
v &= 2A \theta
\end{align*}
\]

Contour lines of \(v\) are 1s to contour lines of \(u\), and give lines of heat flow.
Consider following problem:

\[ \Delta^2 u(x, y) = 0 \quad \text{pt. source} \]

\[ u_x(0, y) = 0 \quad \text{insulated} \]

\[ \text{in 3-D: line source next to insulated wall.} \]

To solve: use extension of method of images.

Consider image problem:

\[ \Delta^2 u = 0 \]

unphysical region

\[ \text{ignore wall} \]
As Laplace's Equation is linear and homogeneous, can add solutions:

Solution of image problem is

\[ u(x,y) = A \ln[(x-a)^2 + y^2] + A \ln[(x+a)^2 + y^2] \]

\[-\infty < x < \infty, \quad -\infty < y < \infty\]

\[(x,y) \neq (\pm a, 0)\]

We claim that this is solution to original problem, including BC, when restrict to physical region

\[ 0 < x < \infty, \quad -\infty < y < \infty. \]

To 'prove', all we have to do is check the BC. (That \( u \) satisfies \( \nabla^2 u(x,y) = 0 \) is evident.)

\[ 0 < x < \infty, \quad x \neq a\]

\[-\infty < y < \infty\]
‘systems biology’ is typically used to indicate the attempt to integrate the huge and complex array of biological data in order to understand the behaviour of biological systems, and to study the relationships and interactions between the various parts of a biological system, such as organelles, cells, physiological systems, organisms etc.

The rapid growing of knowledge at the molecular level (e.g. genomes, transcriptomes, proteomes, metabolomes) is giving, for the first time, the opportunity to constitute the solid ground upon which to create an understanding of living organisms at the system level. Such an effort is meant not only at describing in detail the system structure and behaviour, but also at understanding its reaction in response to external stimuli or disruptions, as in the Systems Biology project [1]. Examples of well-studied and understood molecular networks include gene regulation networks (how genes and their products, proteins and RNAs, are regulating gene expression), metabolic pathways (the network of reactions involving metabolites) and signal transduction cascades (the molecular interactions activating a genetic answer to a signal received from the cell).

Because of the sensitivity and complexity of the data, this new scientific challenge is much more demanding than ever, and is going to involve computer scientists and engineers, mathematicians, physicists, biochemists, and automatic control engineers. Scientists, working in close partnership with the biologists, along this line, fundamental issues like the information management framework, as well as the model construction, analysis and validation phases [2]. Efforts are ongoing, meant to provide for a common and versatile software platform for systems biology research. For instance, in the Systems Biology Workbench project [3], one of the critical issues regard the exchange of data and the interface between software modules. SBML [4, 5] and CELLMIT [6] are modelling languages in systems biology that are aimed at facing such issues.

Although systems biology has no clear end-point, the prize to be attained is immense. From in-silico drug design and testing, to individualised medicine, which will take into account physiological and genetic profiles, there is a huge potential to profoundly affect health care, and medical science in general [7].

Therefore, in this article we propose an agent-oriented conceptual framework to support the modelling and analysis of biological systems. The framework is intended to support life scientists in building models and verifying experimental hypotheses by simulation. We believe that the use of an agent-based computational platform [7, 8] and agent coordination in infrastructures [9], along with the adoption of formal methods [10] will make it possible to harness the complexity of the biological domain by delegating software agents to simulate bio-entities [11, 12, 13]. As by studying the interaction of different models and tools [14], an agent-oriented framework could allow life scientists to analyse and validate system behaviours [15], to verify system’s properties [16], and to design, in a compositional way, new models from known ones. The conceptual framework we propose takes into account the four steps suggested by Kitano in [1]: (i) system structure identification, (ii) system behaviour analysis, (iii) system control, (iv) system design. For each step, our framework exploits agent-oriented
First Problem
\( a = 1, \quad A = -1 \)

\[ \rightarrow \text{heat flow} \]

Second Problem

\[ \rightarrow \text{isotherms} \]

In this case:

\[ u(x, y) = A \ln[(x-a)^2 + y^2] - A \ln[(x+a)^2 + y^2] \]

\[ u(0, y) = 0 \] as required.

\[ v(x, y) = 2A \arctan \left( \frac{2(x-a) y}{x^2 + y^2} \right) \]

Try to show it!

We can do more elaborate image problems:
Can you see for what wedge angles can do it?

**Summary:**

1) Know formula for \( u(x,y) \), point source — be able to show \( \nabla^2 u = 0 \).
2) Understand method of images for point sources/sinks.