MATH2100 Lec. 36 (= MATH2011 Lec. 18)

Other applications of Laplace's Equation

1. Steady irrotational flow of an incompressible fluid. (Potential Flow)

Recall that for an incompressible fluid with velocity field $\underset{\sim}{v}(x,y,z,t)$, conservation of mass $\Rightarrow$

$$(36.1) \quad \boxed{\operatorname{div} \underset{\sim}{v} = 0} \qquad \nabla \cdot \underset{\sim}{v} = 0$$

(see p. 25.3)

i.e. if

$$\underset{\sim}{v}(x,y,z,t) = v_1(x,y,z,t)\underset{\sim}{i} + v_2(x,y,z,t)\underset{\sim}{j} + v_3(x,y,z,t)\underset{\sim}{k}$$

then

$$\boxed{\frac{\partial v_1(x,y,z,t)}{\partial x} + \frac{\partial v_2(x,y,z,t)}{\partial y} + \frac{\partial v_3(x,y,z,t)}{\partial z} = 0} \quad (36.2)$$

For <u>steady</u> flows, we have $\underset{\sim}{v}(x, y, z)$

There is an important class of steady flows, for which $\underset{\sim}{v}(x,y,z)$ can be expressed in the form

$$\boxed{\underset{\sim}{v}(x,y,z) = \underset{\sim}{\nabla}\varphi(x,y,z) \qquad (36.3)}$$

for a suitable scalar field φ.
Thus

$$v_1(x,y,z)\underset{\sim}{i} + v_2(x,y,z)\underline{j} + v_3(x,y,z)\underline{k}$$

$$= \frac{\partial\varphi(x,y,z)}{\partial x}\underset{\sim}{i} + \frac{\partial\varphi(x,y,z)}{\partial y}\underline{j} + \frac{\partial\varphi(x,y,z)}{\partial z}\underline{k}$$

$$\Rightarrow \boxed{v_1(x,y,z) = \frac{\partial\varphi(x,y,z)}{\partial x}, \quad v_2(x,y,z) = \frac{\partial\varphi(x,y,z)}{\partial y}, \quad v_3(x,y,z) = \frac{\partial\varphi(x,y,z)}{\partial z} \qquad (36.4)}$$

A special state of affairs!

To see how special, consider

$$\operatorname{curl} \underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y}\right) - \underline{j}\left(\frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^2 \varphi}{\partial z \partial x}\right)$$
$$+ \underline{k}\left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x}\right)$$

$$= \underset{\sim}{0} \quad \text{if } \varphi \text{ is smooth.}$$

We say such a flow is <u>curl-free</u>, or <u>irrotational</u>.

[Since we also have div $\underset{\sim}{v} = 0$, we can also say flow is <u>div-free</u>, or <u>solenoidal</u>.]

Now look what happens because we have (36.3) + (36.1) :—

$$\underset{\sim}{v} = \underset{\sim}{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\underset{\sim}{i} + \frac{\partial\varphi}{\partial y}\underset{\sim}{j} + \frac{\partial\varphi}{\partial z}\underset{\sim}{k}$$

$$0 = \underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$= \nabla^2\varphi$$

Thus the scalar field φ satisfies <u>Laplace's Equation</u>

$$\boxed{\nabla^2\varphi(x,y,z) = 0} \qquad (36.5)$$

We call $\varphi(x,y,z)$ the <u>velocity potential</u> and talk of <u>potential flow</u>.

In 2-D (no z-dependence): -

$$\boxed{\begin{aligned} \underset{\sim}{v}(x,y) &= v_1(x,y)\,\underset{\sim}{i} + v_2(x,y)\,\underset{\sim}{j} \\ &= \frac{\partial \varphi(x,y)}{\partial x}\,\underset{\sim}{i} + \frac{\partial \varphi(x,y)}{\partial y}\,\underset{\sim}{j} \\ \text{where} \quad & \varphi_{xx}(x,y) + \varphi_{yy}(x,y) = 0 \end{aligned}} \tag{36.6}$$

We now know from our experience with heat conduction that in this situation we can find another scalar field $\psi(x,y)$, say (the harmonic function conjugate to $\varphi(x,y)$) with

$$\boxed{\frac{\partial \varphi(x,y)}{\partial x} = \frac{\partial \psi(x,y)}{\partial y}, \quad \frac{\partial \varphi(x,y)}{\partial y} = -\frac{\partial \psi(x,y)}{\partial x}} \tag{36.7}$$

(Cauchy-Riemann equations)

and

$$\boxed{\nabla^2 \psi(x,y) \equiv \psi_{xx}(x,y) + \psi_{yy}(x,y) = 0} \tag{36.8}$$

We also know that the level (contour) lines of ψ are field lines of $\underset{\sim}{\nabla}\varphi$.

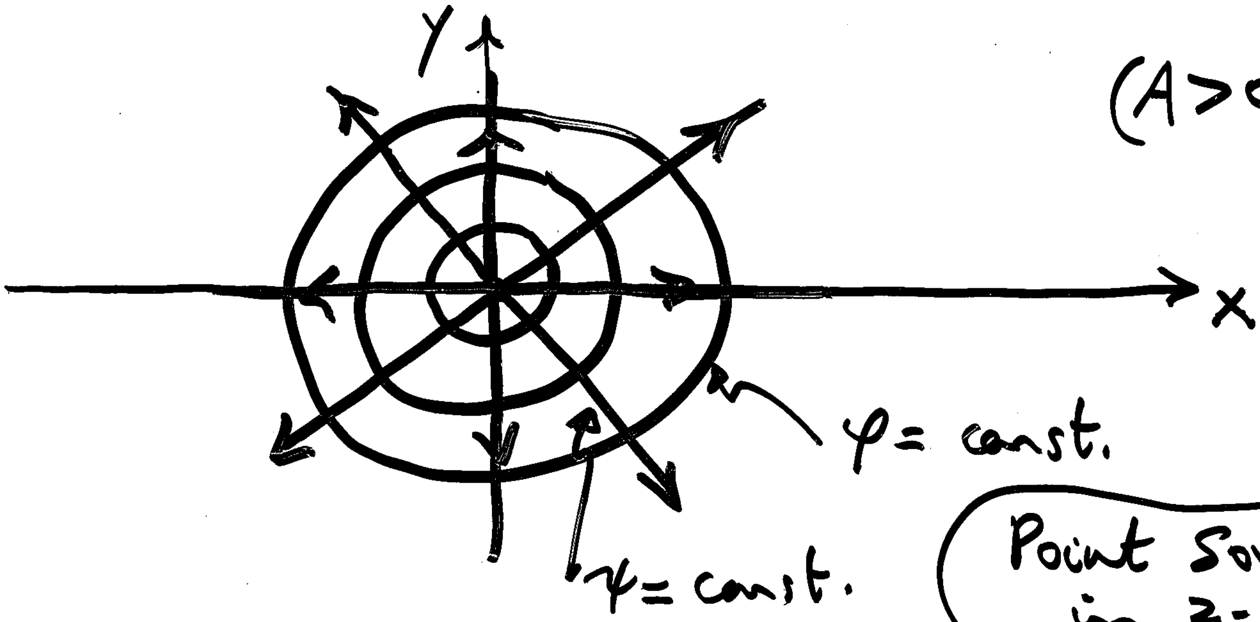
But $\underset{\sim}{\nabla}\varphi = \underset{\sim}{v}$!

So level lines of ψ are field lines of velocity field $\underset{\sim}{v}$ – the lines along which fluid is flowing – called the <u>streamlines</u> of the flow. For this reason ψ is called the velocity stream function, or simply, the <u>stream function</u>.

<u>EX</u>:

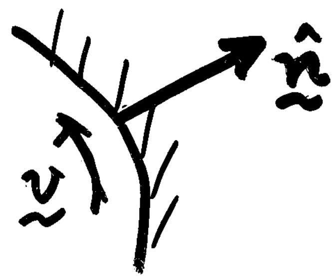
$$\varphi(x,y) = A \ln(x^2+y^2), \quad (x,y) \neq (0,0)$$

$(A>0)$



Point Source in 2-D

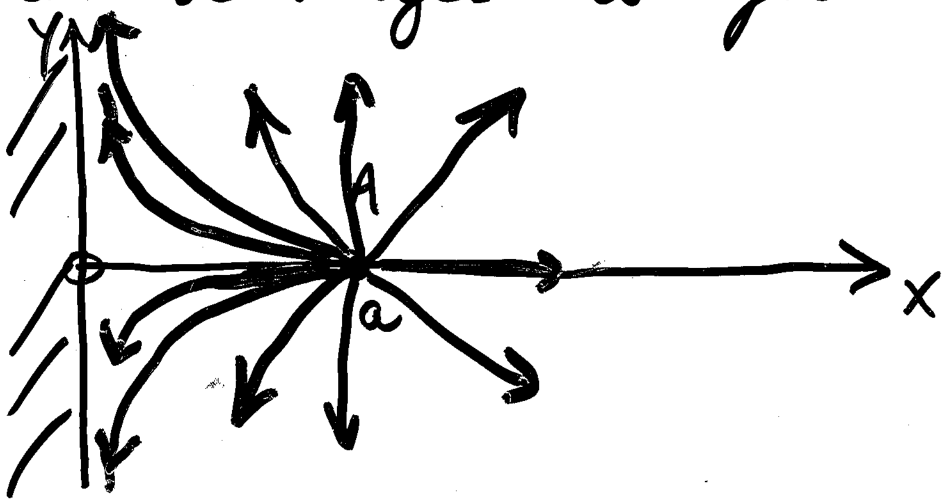
At an impenetrable wall or boundary:



$$\underset{\sim}{v} \cdot \hat{\underset{\sim}{n}} = 0$$

[$\underline{J} = \rho_0 \underline{v}$ is mass flux vector. Condition says no flux of mass across wall.]

Then could use images to get

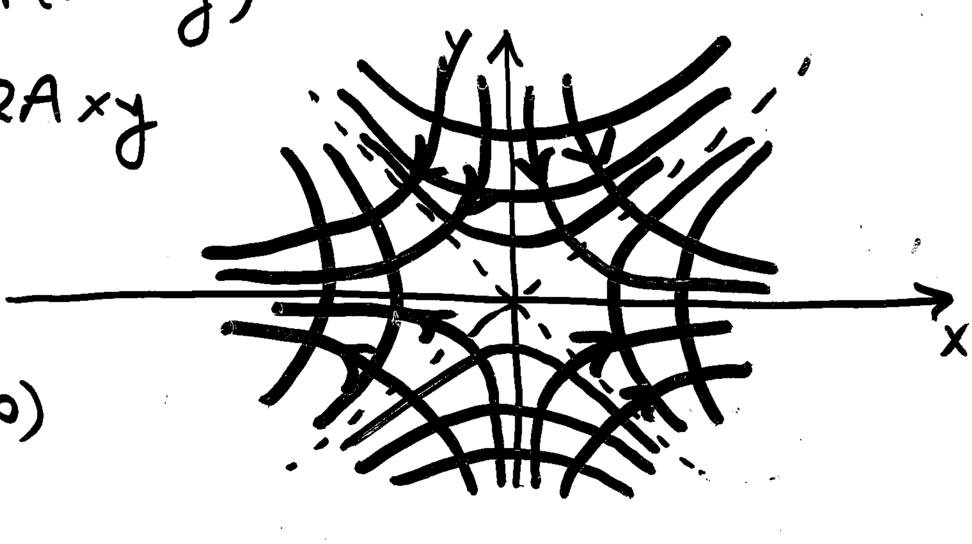


and so on – same maths as before.

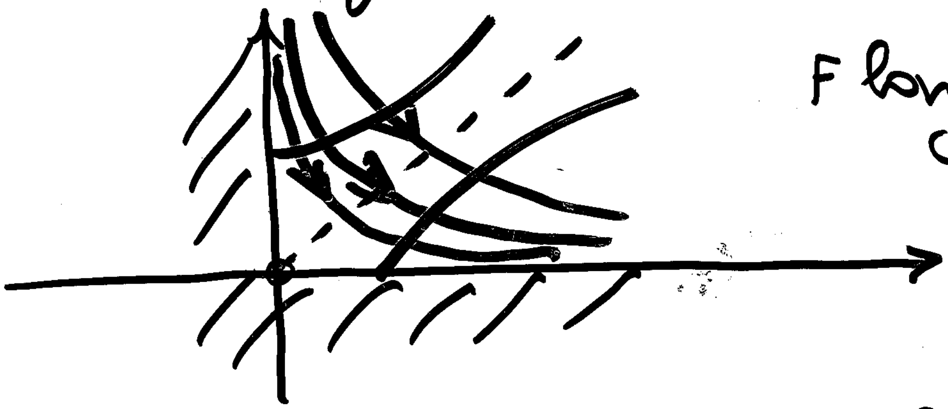
Another <u>EX</u>: (cf. p. 34.8)

$$\varphi(x,y) = A(x^2 - y^2)$$
$$\psi(x,y) = 2Axy$$

(A>0)



An unlikely looking flow pattern. But note can put an impenetrable wall along any streamline without changing pattern. So we get solution for:



Flow in a corner

$$\left.\begin{aligned} \varphi(x,y) &= A(x^2-y^2) \\ \underset{\sim}{v} = \underset{\sim}{\nabla}\varphi &= 2Ax\,\underset{\sim}{i} - 2Ay\,\underset{\sim}{j} \end{aligned}\right\} \quad \begin{aligned} x&>0 \\ y&>0 \end{aligned}$$

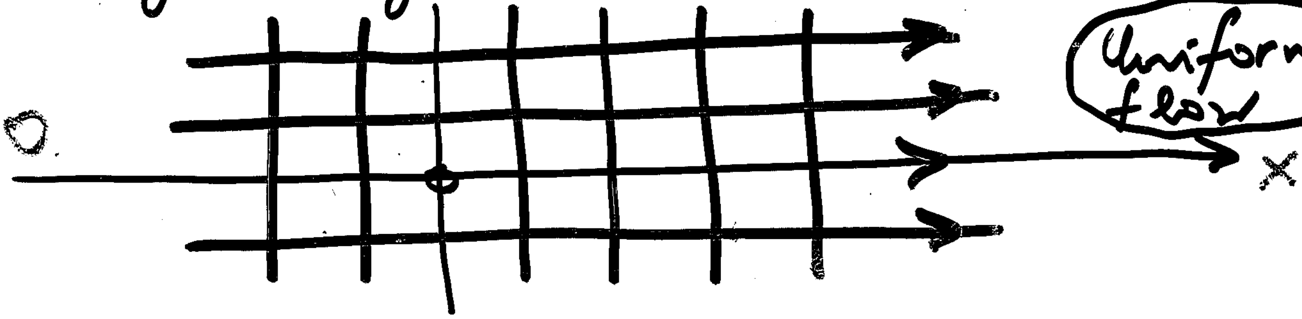
Simplest example of all:

$$\varphi(x,y) = Bx \quad (\Rightarrow \underline{v} = B\underset{\sim}{i})$$

$$(\Rightarrow \varphi_{xx} = 0, \ \varphi_{yy} = 0 \ \Rightarrow \ \nabla^2\varphi = 0)$$

$$\varphi_x = \psi_y \text{ and } \varphi_y = -\psi_x \ \Rightarrow \ \psi(x,y) = By$$

$B > 0$



Uniform flow

2. Potential theory in Electrostatics

The variation of the electromagnetic field in free space (outside or away from all charges) is governed by (free) Maxwell's Equations:

Electric field $\underset{\sim}{E}(x, y, z, t)$
Magnetic induction $\underset{\sim}{B}(x, y, z, t)$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 0 \qquad \underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{\partial B}{\partial t}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \qquad \underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{1}{c^2} \frac{\partial E}{\partial t}$$

In static situations (no t-dependence) the equations decouple:

$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 0, \ \underset{\sim}{\nabla} \times \underset{\sim}{E} = \underset{\sim}{0}$ (36.9) Equations of electrostatics	$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0, \ \underset{\sim}{\nabla} \times \underset{\sim}{B} = \underset{\sim}{0}$ (36.10) Equations of magnetostatics

Electrostatics

As for irrotational, incompressible flow, we can introduce $\varphi(x,y,z)$ such that

$$\underset{\sim}{E}(x,y,z) = -\underset{\sim}{\nabla}\varphi(x,y,z) \quad \left(\Rightarrow \underset{\sim}{\nabla}\times\underset{\sim}{E} = \underset{\sim}{0} \text{ as required} - \text{cf. p. } 36.3\right)$$

convention

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = 0$$

$$\Rightarrow -\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\varphi = -\nabla^2\varphi = 0$$

So again we come to Laplace's Equation:

$$\nabla^2\varphi(x,y,z) \equiv \frac{\partial^2\varphi(x,y,z)}{\partial x^2} + \frac{\partial^2\varphi(x,y,z)}{\partial y^2} + \frac{\partial^2\varphi(x,y,z)}{\partial z^2} = 0$$

This time, $\varphi(x,y,z)$ is called the <u>electrostatic potential</u>.

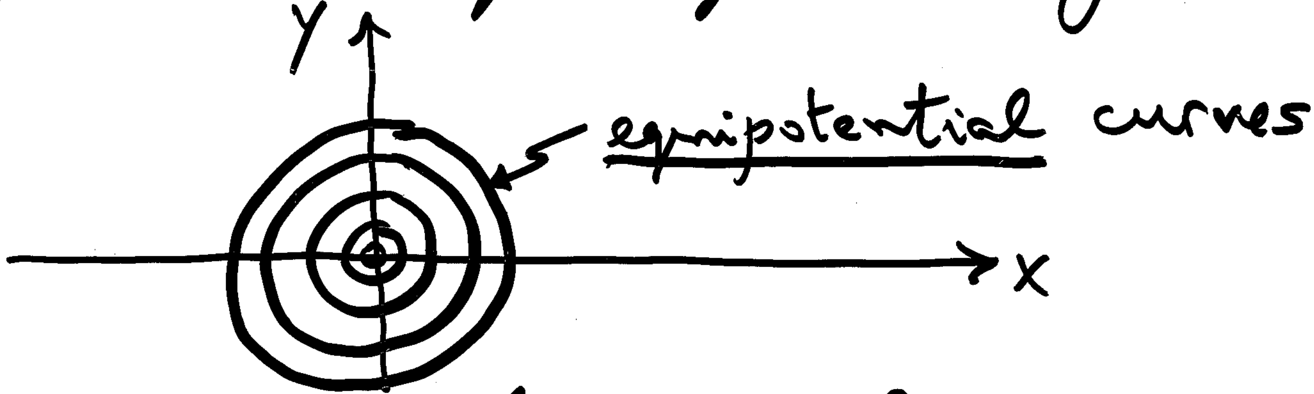
In 2-D (no z-dependence) we have the same maths as before:

$$\varphi_{xx}(x,y) + \varphi_{yy}(x,y) = 0.$$

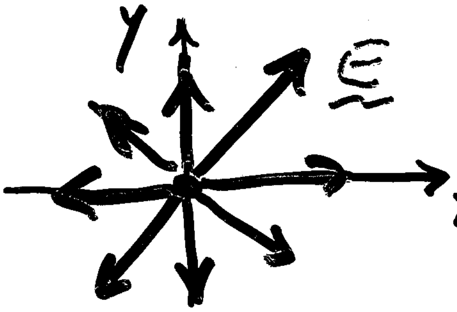
Thus we have, for example

$$\varphi(x,y) = A \ln(x^2+y^2) \qquad (x,y) \neq (0,0)$$

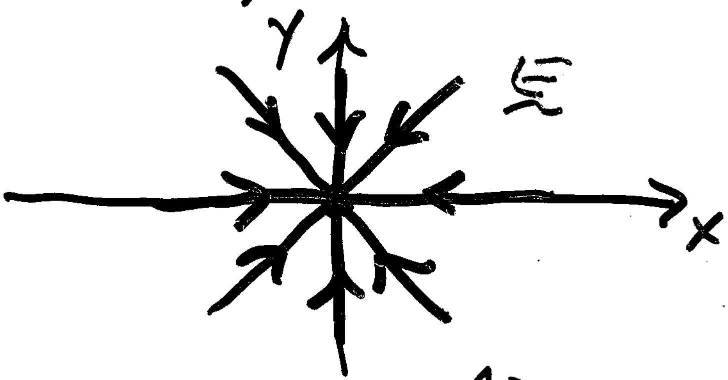
- point source of charge at origin:



The conjugate harmonic function now gives the field lines of $\underline{E}$.
("E-lectrostatic stream function"?)



$A<0$
(positive charge)



$A>0$
(negative charge)

and so on ...

Summary:

1) Laplace's Equation also important for description of a class of fluid flows – potential flows.

2) Understand equations relating $\underset{\sim}{v}(x,y)$, $\varphi(x,y)$, $\psi(x,y)$ and the interpretation of these fields

3) Again, Laplace's Equation important in electrostatics.

4) Understand equations relating $\underset{\sim}{E}(x,y)$, $\varphi(x,y)$

5) Appreciate the unifying power of the mathematics!

K ~~p 459, 812-814, 798-803~~
p. 412, 761-763, 750-753