**Lec. 3** **MATH2100/2010**

We were considering

$$y''(t) + 2y'(t) - 15y(t) = 0$$

Let $y_1 = y$, $y_2 = y'$

$$\Rightarrow (*)\ \underset{\sim}{y}'(t) = A\underset{\sim}{y}(t), \quad A = \begin{bmatrix} 0 & 1 \\ 15 & -2 \end{bmatrix}, \quad \underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Try $\underset{\sim}{y}(t) = \underset{\sim}{x} e^{\lambda t}$, $\underset{\sim}{x} = \begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\Rightarrow \underset{\sim}{y}'(t) = \lambda \underset{\sim}{x} e^{\lambda t}\right)$$

$$(*) \Rightarrow \lambda \underset{\sim}{x} e^{\lambda t} = A\underset{\sim}{x} e^{\lambda t}$$

$$\Rightarrow A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\Rightarrow (A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 0-\lambda & 1 \\ 15 & -2-\lambda \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(3.2)

But the equation $B\underset{\sim}{x} = \underset{\sim}{0}$ has a <u>nonzero</u> solution $\underset{\sim}{x} \neq \underset{\sim}{0}$ <u>if and only if</u> the matrix $B$ is <u>singular</u> (has <u>no inverse</u> $B^{-1}$)

(see if $B^{-1}$ <u>did</u> exist, could do

$$B\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \underbrace{B^{-1}B}_{= I}\underset{\sim}{x} = B^{-1}\underset{\sim}{0} \Rightarrow \underset{\sim}{x} = \underset{\sim}{0}\ )$$

But the condition for $B$ to be singular is $\det(B) = 0$ (See K §4.0)

Since we are only interested in nonzero solutions $\underset{\sim}{x} \neq \underset{\sim}{0}$ of $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$, we must require

$$\det(A - \lambda I) = 0$$

Eigenvalue condition ~~(or equation)~~ for $A$

So
$$\begin{vmatrix} 0-\lambda & 1 \\ 15 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-\lambda)(-2-\lambda) - 15 = 0$$
$$\Rightarrow \quad \lambda^2 + 2\lambda - 15 = 0 \quad \text{(charac. quadratic)}$$
$$\Rightarrow \quad (\lambda+5)(\lambda-3) = 0$$
$$\Rightarrow \quad \lambda_1 = 3, \quad \lambda_2 = -5$$

So the roots of the characteristic quadratic now appear as the <u>eigenvalues of A</u>.

Eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -5$ (3.4) = (2.14)

Finding e'vectors: $\underline{\lambda_1 = 3}$:

$$(A - 3I)\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \quad \begin{bmatrix} 0-3 & 1 \\ 15 & -2-3 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} -3u + v &= 0 \\ 15u - 5v &= 0 \end{aligned}\right\} \Rightarrow \quad v = 3u$$

Choosing $u = 1$, $\underline{\text{e'vector is}}$ $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\underline{\lambda_2 = -5}$:

$$(A + 5I)\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \quad \begin{bmatrix} 5 & 1 \\ 15 & 3 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} 5u + v &= 0 \\ 15u + 3v &= 0 \end{aligned}\right\} \Rightarrow \quad v = -5u$$

Choosing $u = 1$, $\underline{\text{e'vector is}}$ $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

So now know that $\underline{x}^{(1)} e^{3t}$ and $\underline{x}^{(2)} e^{-5t}$ are solutions.

3.5 = 2.15

Then general solution is

$$\underset{\sim}{y}(t) = \alpha \underset{\sim}{x}^{(1)} e^{3t} + \beta \underset{\sim}{x}^{(2)} e^{-5t}$$

i.e. $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + \beta \begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-5t}$

[Recall $y(t) = y_1(t)$ — top component

$= \alpha e^{3t} + \beta e^{5t}$ — as before

Also $\underset{\sim}{y_2(t) = y'(t) = 3\alpha e^{3t} - 5\beta e^{-5t}}$

With ICs $y(0) = -1$ $(\Rightarrow y_1(0) = -1)$

$y'(0) = 13$ $(\Rightarrow y_2(0) = 13)$

we get

$$\underset{\sim}{y}(0) = \begin{pmatrix} -1 \\ 13 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$\Rightarrow \alpha = 1, \ \beta = -2$ as before,

So get unique solution:

$$\underset{\sim}{y}(t) = \underset{\sim}{x}^{(1)} e^{3t} - 2 \underset{\sim}{x}^{(2)} e^{-5t}$$

3.6

**EX:**

$$y''(t) + 4y(t) = 0$$

a) **Directly:**

Try $y(t) = e^{\lambda t}$

$\Rightarrow \quad (\lambda^2 + 4)e^{\lambda t} = 0$

$\Rightarrow \quad \lambda = \pm 2i$

General solution:

$$y(t) = Ae^{2it} + Be^{-2it}$$ Complex form

$$= A[\cos(2t) + i\sin(2t)] + B[\cos(2t) - i\sin(2t)]$$

$$= C\cos(2t) + D\sin(2t)$$ real form

$C = (A+B) \quad D = i(A-B)$

Use ICs to fix $A, B$ [or $C, D$]

b) <u>As a system:</u> 3.7

Let $y_1 = y$ , $y_2 = y'$

Then get: $y_1' = y' = y_2$

$y_2' = y'' = -4y = -4y_1$

i.e.

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

or

$$\underset{\sim}{y}'(t) = A\,\underset{\sim}{y}(t) \qquad A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$$

Try $\underset{\sim}{y}(t) = \underset{\sim}{x}\, e^{\lambda t}$ $\left(\Rightarrow \underset{\sim}{y}'(t) = \lambda \underset{\sim}{x}\, e^{\lambda t}\right)$

$\Rightarrow \quad \lambda \underset{\sim}{x}\, e^{\lambda t} = A \underset{\sim}{x}\, e^{\lambda t}$

$\Rightarrow \quad A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ eigenvalue equation for A

(3.8)

So $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

<u>Eigenvalue condition</u>:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ -4 & 0-\lambda \end{vmatrix}$$

$$= (-\lambda)(-\lambda) + 4$$

So

$$\lambda^2 + 4 = 0$$

$\Rightarrow \quad \lambda_1 = 2i \quad , \quad \lambda_2 = -2i \quad$ e'values

<u>Finding e'vectors</u>: $\lambda_1 = 2i$:

$$(A - 2i\,I)\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} 0-2i & 1 \\ -4 & 0-2i \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{array}{l} -2iu + v = 0 \\ -4u - 2iv = 0 \end{array}\right\} \Rightarrow v = 2iu$$

Choosing $u = 1$ we get $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$

Similarly $(i \leftrightarrow -i)$ $\quad \underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$
corresponds to e'value $\lambda_2 = -2i$

Now have that each of $\underset{\sim}{x}^{(1)} e^{2it}$ and $\underset{\sim}{x}^{(2)} e^{-2it}$ is a solution, and general solution is:-

$$\underset{\sim}{y}(t) = c_1 \underset{\sim}{x}^{(1)} e^{2it} + c_2 \underset{\sim}{x}^{(2)} e^{-2it}$$

$$= c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it}$$

general solution in complex form

Getting general solution in <u>real form</u>:

$$\underset{\sim}{y}(t) = \begin{pmatrix} c_1\left[\cos(2t) + i\sin(2t)\right] \\ 2ic_1\left[\cos(2t) + i\sin(2t)\right] \end{pmatrix} + \begin{pmatrix} c_2\left[\cos(2t) - i\sin(2t)\right] \\ -2ic_2\left[\cos(2t) - i\sin(2t)\right] \end{pmatrix}$$

$$= \begin{pmatrix} (c_1+c_2)\cos(2t) + i(c_1-c_2)\sin(2t) \\ -2(c_1+c_2)\sin(2t) + 2i(c_1-c_2)\cos(2t) \end{pmatrix}$$

$$= d_1\begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + d_2\begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}$$

where $d_1 = c_1 + c_2$, $d_2 = i(c_1 - c_2)$

Fix $d_1$, $d_2$ (or $c_1$, $c_2$) using ICs.

A <u>general system</u> of <u>$n$</u> coupled 1st order ODEs has:

$n$ unknowns $y_1(t), y_2(t), \cdots y_n(t)$ satisfying equations of the form:

$$\left.\begin{aligned} y_1'(t) &= f_1(t, y_1(t), y_2(t), \dots, y_n(t)) \\ y_2'(t) &= f_2(t, y_1(t), y_2(t), \dots, y_n(t)) \\ &\vdots \\ y_n'(t) &= f_n(t, y_1(t), y_2(t), \dots, y_n(t)) \end{aligned}\right\} \quad (3.1)$$

where

$f_1, f_2, f_3, \cdots, f_n$ are given.

OR, equivalently:

$\underset{\sim}{y}'(t) = \underset{\sim}{F}(t, \underset{\sim}{y}(t))$ where

$$\underset{\sim}{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad \underset{\sim}{F}(t, \underset{\sim}{y}(t)) = \begin{pmatrix} f_1(t, \underset{\sim}{y}(t)) \\ f_2(t, \underset{\sim}{y}(t)) \\ \vdots \\ f_n(t, \underset{\sim}{y}(t)) \end{pmatrix}$$

A solution for $a<t<b$ has the form

$$y_1(t)=h_1(t),\ y_2(t)=h_2(t),\ \dots\ ,y_n(t)=h_n(t)$$

where $h_1, h_2, \dots, h_n$ are functions that are once-differentiable, for $a<t<b$.

i.e.

$$\underset{\sim}{y}(t) = \underset{\sim}{h}(t)\ , \quad \underset{\sim}{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

A typical initial-value problem (IVP) consists of (**) plus ICs at some $t_0$:

$$y_1(t_0) = K_1,\ y_2(t_0) = K_2,\ \dots\ y_n(t_0) = K_n$$

i.e.

$$(3.2)\quad \underset{\sim}{y}(t_0) = \underset{\sim}{K}\ , \quad \underset{\sim}{K} = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{pmatrix} \ \text{(constant vector) - given}$$

Here we can suppose $a < t_0 < b$, and we are only interested in the solution for $t_0 \leq t < b$

For such an IVP, there <u>exists</u> a <u>unique</u> solution, <u>provided</u> the functions $f_1, f_2, \cdots, f_n$ are <u>suitably smooth</u>.

(See K p. ~~154~~ 137 Theorem 1)

You do not need to know the Theorem, but you should know ~~that~~ things can go wrong if the $f$'s are not nice!

(3.14)

<u>Summary</u>:

~~K §3.0, §3.1~~ K: 4.0, 4.1

1) Know how to solve 2nd order, linear ODE with constant coefficients in two ways:

a) Directly

b) By converting to a system, then finding e'values and e'vectors

2) In case that e'values are complex

a) know how to use ($i \leftrightarrow -i$) trick to save work.

b) know how to go from complex form to real form of solution

3) Read and understand nature of existence + uniqueness <u>Theorem !, K p. ~~159~~ 137</u>