Lec 4: MATH2100/2010

There exists a unique solution of the IVP

$$\underset{\sim}{y}'(t) = \underset{\sim}{F}(t, \underset{\sim}{y}(t)) \quad , \quad \underset{\sim}{y}(0) = \underset{\sim}{K}$$

provided the components $f_1\ f_2 \cdots f_n$ of $\underset{\sim}{F}$ are 'suitably smooth' functions.

(K p ~~159~~ 137 Theorem 1)

Things go wrong if $\underset{\sim}{F}$ not 'suitably smooth':

EX:

$$\begin{cases} y_1' = \sqrt[4]{\dfrac{|(y_1+2)(y_2-1)|}{2}} \\ y_2' = 2\sqrt{|y_1+2|} \end{cases}$$

ICs: $y_1(0) = -2$, $y_2(0) = 1$

There exist (at least) two solutions:

1) $y_1(t) = -2$, $y_2(t) = 1$, $-\infty < t < \infty$

2) $y_1(t) = \begin{cases} \dfrac{t^2}{4} - 2, & t \geq 0 \\ -\dfrac{t^2}{4} - 2, & t < 0 \end{cases}$ $y_2(t) = \begin{cases} \dfrac{t^2}{2} + 1, & t \geq 0 \\ -\dfrac{t^2}{2} + 1, & t < 0 \end{cases}$

Check!

EX: $\begin{cases} y_1' = y_2 \\ y_2' = \dfrac{6y_1}{t^2} \end{cases}$ ICs : $y_1(0) = 6$, $y_2(0) = 6$

No solution exists.

[General solution of system is

$y_1(t) = \alpha t^3 + \beta t^{-2}$, $y_2(t) = 3\alpha t^2 - 2\beta t^{-3}$]

EX: $\begin{cases} y_1' = y_2 \\ y_2' = (y_1)^6 + e^t \end{cases}$ ICs : $y_1(0) = 0$, $y_2(0) = 0$

Theorem shows that there exists a unique solution in this case. Unfortunately, it cannot be expressed in terms of known functions

The simplest systems of ODEs are <u>linear</u>

For $n=2$:

$$\begin{cases} y_1'(t) = a_{11}(t)\,y_1(t) + a_{12}(t)\,y_2(t) + g_1(t) \\ y_2'(t) = a_{21}(t)\,y_1(t) + a_{22}(t)\,y_2(t) + g_2(t) \end{cases}$$

Two coupled, first order, linear ODEs

Here $a_{11}(t)$, $a_{12}(t)$, $a_{21}(t)$, $a_{22}(t)$, $g_1(t)$, $g_2(t)$ are <u>given</u> functions.

Can write as $\quad \underset{\sim}{y}'(t) = A(t)\,\underset{\sim}{y}(t) + \underset{\sim}{g}(t)$

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}, \quad \underset{\sim}{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \underset{\sim}{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

given (<u>known</u>) $\qquad$ <u>unknown</u> $\qquad$ given (<u>known</u>)

[For general n, $\underset{\sim}{y}(t)$ would be an n-vector, $A(t)$ would be $n \times n$, $\underset{\sim}{g}(t)$ would be an n-vector]

For a linear system with ICs $\underset{\sim}{y}(t_0) = \underset{\sim}{y}_0$, (const.)
there exists a unique solution if all the $a_{ij}(t)$ and $g_i(t)$ are continuous (at and near t_0). (<u>K p. 138 Theorem 2</u>).

We shall <u>mainly</u> deal with linear systems

$$\underset{\sim}{y}'(t) = A(t)\,\underset{\sim}{y}(t) + \underset{\sim}{g}(t),$$

<u>mainly</u> with $n=2$, <u>mainly</u> in the <u>homogeneous case</u> $\underset{\sim}{g}(t) = \underset{\sim}{0}$, and mainly in the case $A =$ <u>const.</u> 2×2 matrix!

[Then IVP <u>always</u> has a unique solution!]

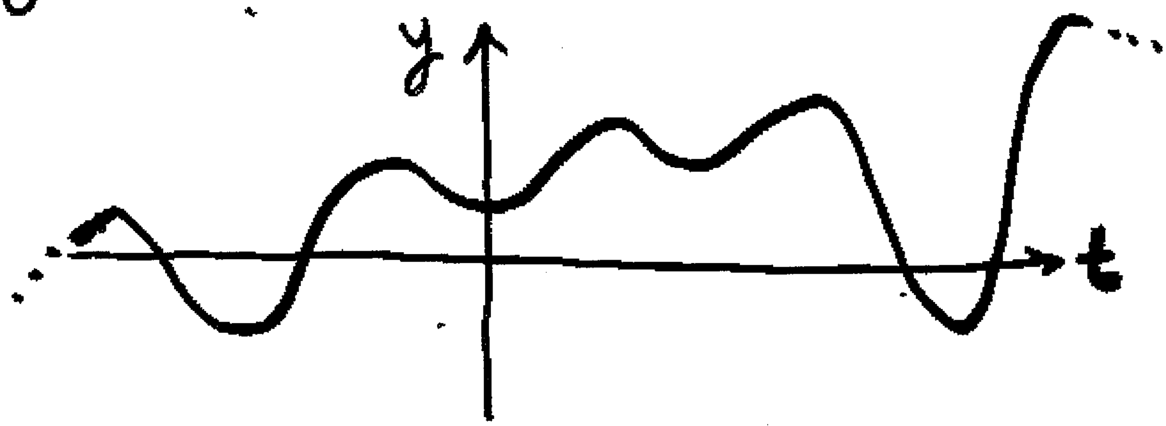
So:

$$(4.1) \qquad \underset{\sim}{y}'(t) = A\,\underset{\sim}{y}(t), \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(a_{11}, a_{12} not necess. equal to 0, 1 now!)

ICs: $\underset{\sim}{y}(0) = \underset{\sim}{y}_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}$

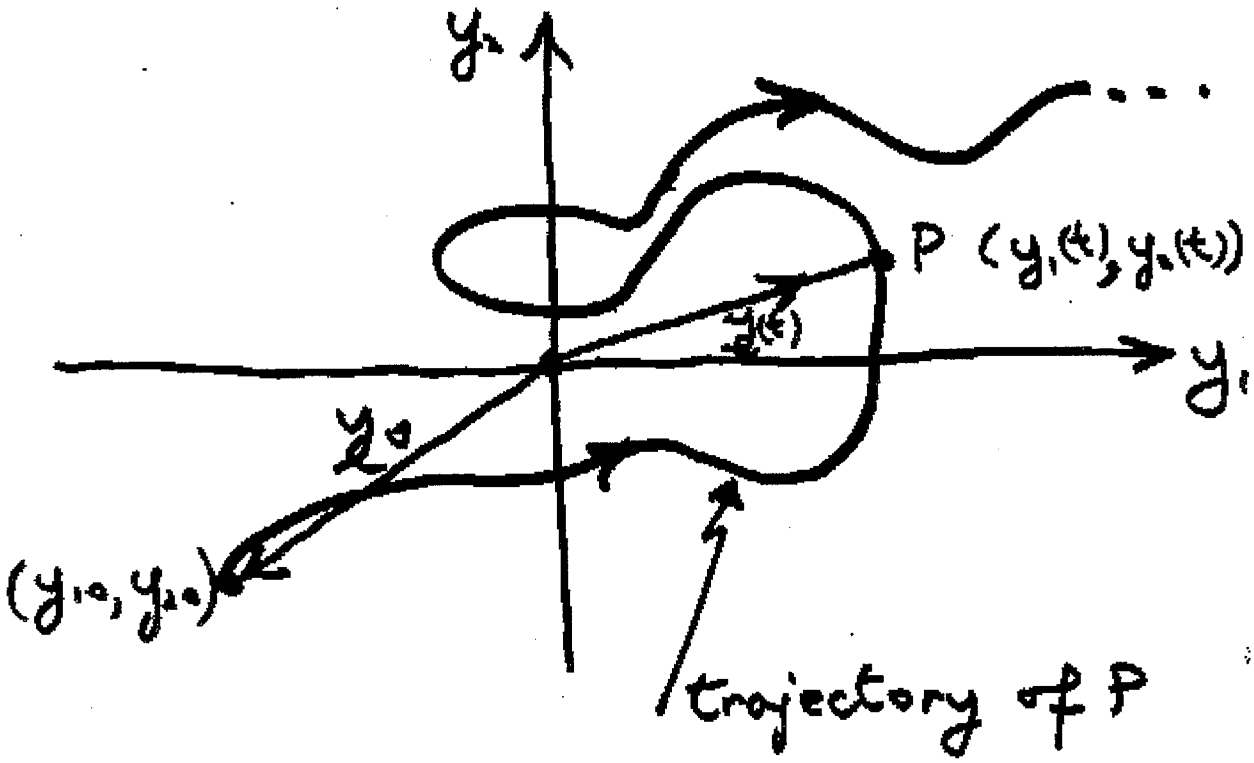
Recall for ODE $y''(t) = F(t, y(t), y'(t))$, to visualize solution we draw graph of (or get computer to draw graph of) $y(t)$ versus t.



What to do to visualize solution of (4.1)?

Could draw graphs of $y_1(t)$ v. t and $y_2(t)$ v. t

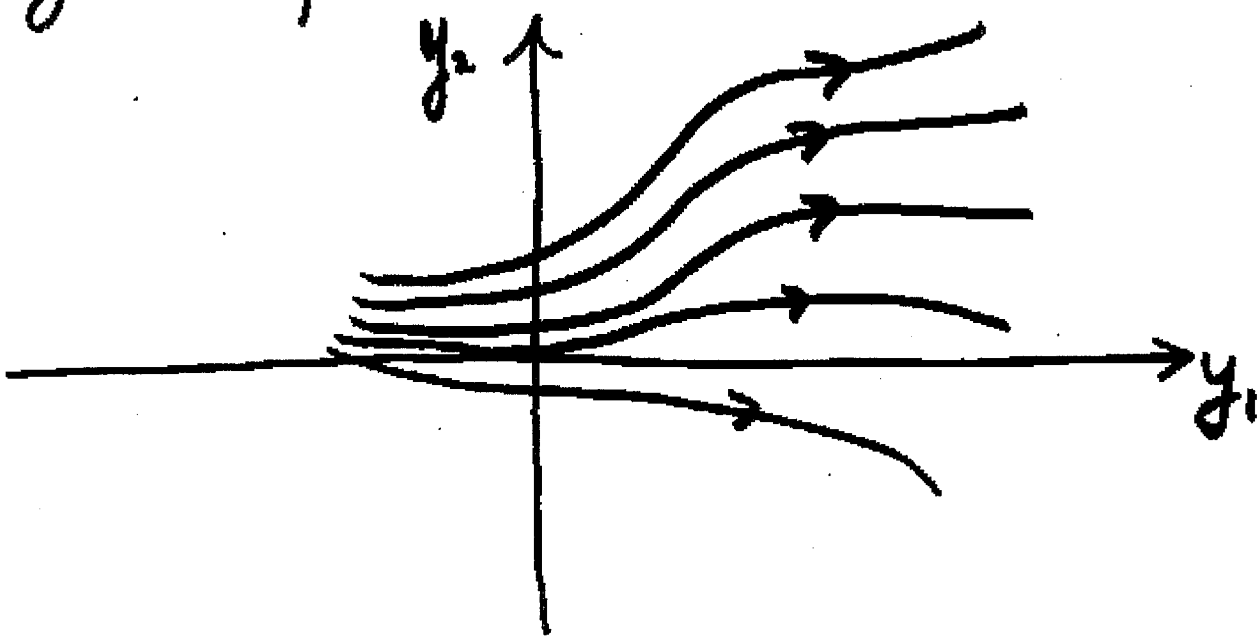
A more interesting thing is to consider $(y_1(t), y_2(t))$ as <u>coordinates</u> of a point P moving with time in the $y_1 y_2$-plane, starting at (y_{10}, y_{20}) at time t_0:



Arrows on trajectory of P indicate direction of increasing time t.

(See $\underset{\sim}{y}(t)$ is <u>position vector</u> of P at time t.)

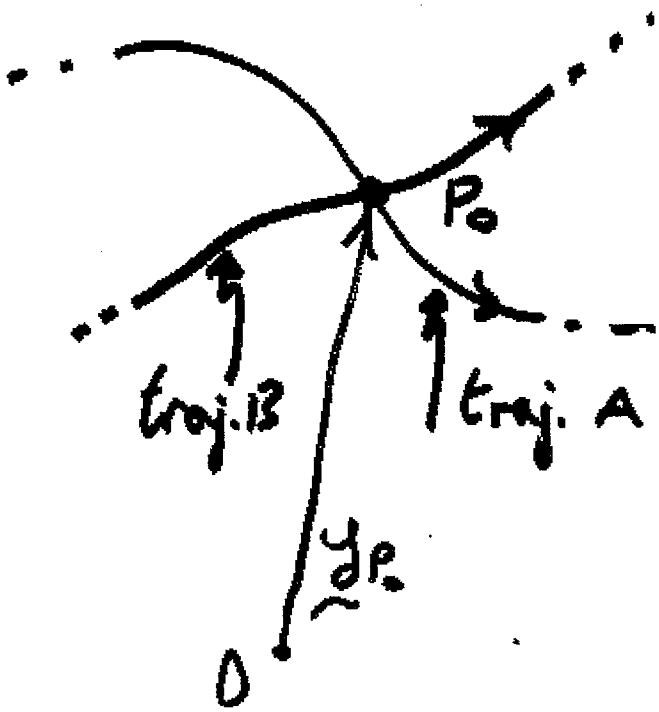
If we consider <u>many different ICs</u>, leading to <u>many different trajectories</u>, we build up a <u>phase-portrait</u> of the system of ODEs:



The $y_1 y_2$-plane is usually called

the <u>phase-plane</u>

<u>Note</u>: Two trajectories <u>cannot cross</u> (and a trajectory cannot cross itself) at a finite value of t.



<u>Pf</u>:

Suppose

$$\underset{\sim}{y}_A(t_A) = \underset{\sim}{y}_{P_0}$$

$$\underset{\sim}{y}_B(t_B) = \underset{\sim}{y}_{P_0}$$

Let $\underset{\sim}{y}(t) = \underset{\sim}{y}_A(t+t_A) - \underset{\sim}{y}_B(t+t_B)$

Then $\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t)$ and $\underset{\sim}{y}(0) = \underset{\sim}{0}$ <u>Check!</u>

But one solution of these equations is $\underset{\sim}{y}(t) = \underset{\sim}{0}$

By uniqueness theorem, $\underset{\sim}{y}(t) = \underset{\sim}{0}$

$$\Rightarrow \underset{\sim}{y}_A(t+t_A) = \underset{\sim}{y}_B(t+t_B)$$

$\Rightarrow$ traj_A and traj_B 'look same' <u>near</u> P_0 — do not cross.

For a 2-component system

(4.1) $\quad \underline{y}'(t) = A\,\underline{y}(t) \quad A = \underset{2\times 2}{\text{const.}}$

there are <u>6 types of phase-portrait</u> <u>that</u> can arise. We will now go through them one-by-one.

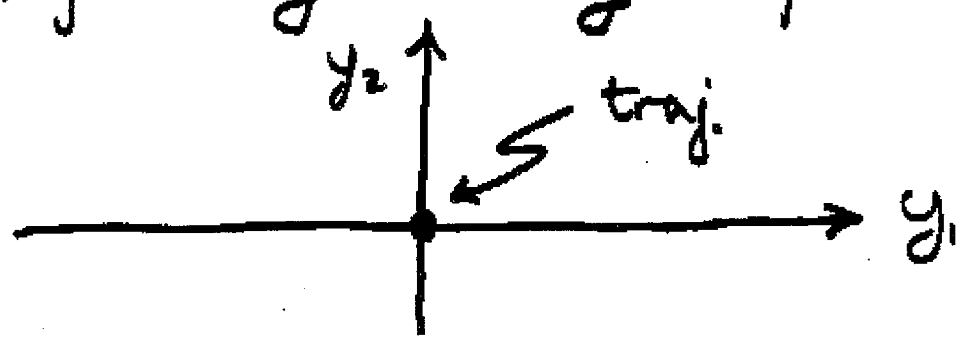
<u>Note firstly</u>: $\underset{\sim}{y}(t) = \underset{\sim}{0}$ is <u>always</u>

a solution, however A looks.

= The <u>trivial solution</u>.

= The unique solution of (4.1) with ICs $\underset{\sim}{y}(t_0) = \underset{\sim}{0}$, any t_0.

This trajectory is easy to plot!



Type 1 : Improper Node

(A has two real e'values with same sign.)

Ex: $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

$(*)\ \underset{\sim}{y}'(t) = A \underset{\sim}{y}(t) \iff \begin{cases} y_1' = \frac{3}{2} y_1 + \frac{1}{2} y_2 \\ y_2' = \frac{1}{2} y_1 + \frac{3}{2} y_2 \end{cases}$

Try $\underset{\sim}{y}(t) = \underset{\sim}{x} e^{\lambda t}$, $\quad \underset{\sim}{x} = \begin{pmatrix} u \\ v \end{pmatrix}$ (const.)

$(\Rightarrow \underset{\sim}{y}'(t) = \lambda \underset{\sim}{x} e^{\lambda t})$

$(*) \Rightarrow \quad A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$\Rightarrow \quad (A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow$ Eigenvalue condition:

$$0 = \det(A - \lambda I) = \begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix}$$

$$= \left(\tfrac{3}{2} - \lambda\right)^2 - \tfrac{1}{4} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

Eigenvalues: $\lambda_1 = 1$, $\quad \lambda_2 = 2$

Summary:

1) Understand ideas of existence and uniqueness (K pp ~~159, 160~~ 137, 138, Theorems 1, 2) – no need to know details.

2) Understand idea of phase-plane and trajectories. (→ phase-portrait)

3) Trajectories never cross.

4) $\underset{\sim}{y}(t) = \underset{\sim}{0}$ is always a solution of $\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t)$ whatever A is. Trajectory is the point at origin O in $y_1 y_2$-plane.

K pp ~~159, 160~~ 137, 138, §~~3.0, 3.1, 3.2~~ 4.0, 4.1, 4.2