Lec. 5 MATH 2100/2010

(5.1) $\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t)$ $\qquad A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$ (see p. 4.11)

Eigenvectors: $\lambda_1 = 1$:

$$(A - \lambda_1 I)\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \begin{bmatrix} \frac{3}{2}-1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2}-1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} \frac{1}{2}u + \frac{1}{2}v = 0 \\ \frac{1}{2}u + \frac{1}{2}v = 0 \end{array}\right\} \Rightarrow v = -u$$

Choosing $u = 1$, we get $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 2$:

$$(A - \lambda_2 I)\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \begin{bmatrix} \frac{3}{2}-2 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2}-2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} -\frac{1}{2}u + \frac{1}{2}v = 0 \\ \frac{1}{2}u - \frac{1}{2}v = 0 \end{array}\right\} \Rightarrow v = u$$

Choosing $u = 1$, we get $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Now each of $\underset{\sim}{x}^{(1)} e^{t}$ and $\underset{\sim}{x}^{(2)} e^{2t}$ is a solution.

General solution of (5.1) is:—

$$\underset{\sim}{y}(t) = A \underset{\sim}{x}^{(1)} e^{t} + B \underset{\sim}{x}^{(2)} e^{2t}$$

$$= A \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + B \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

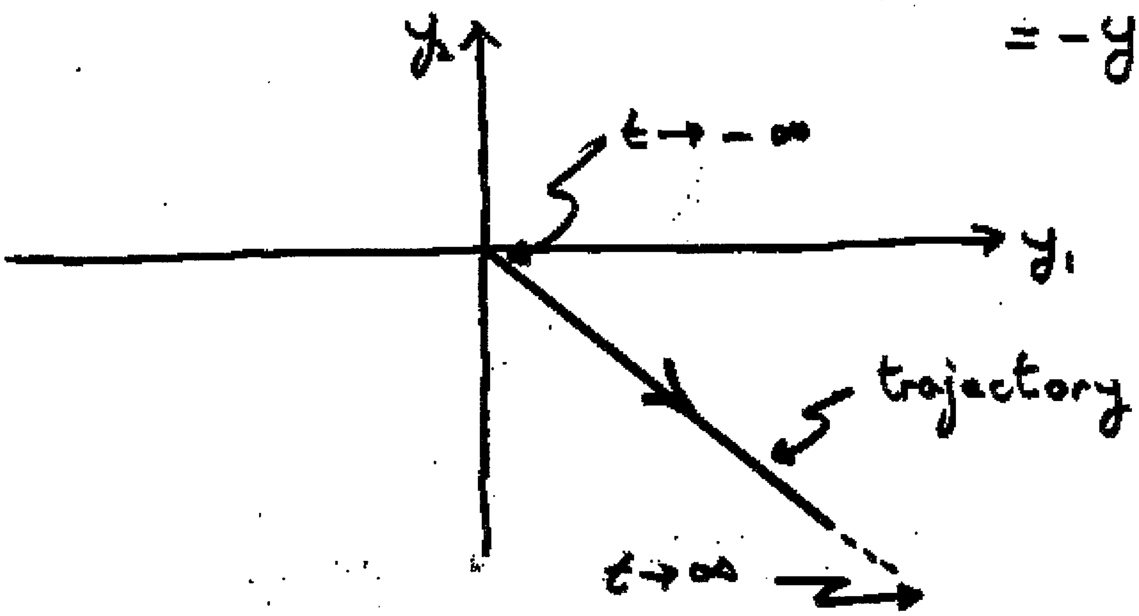
Each choice of ICs, or equivalently, each choice of A and B, defines a trajectory in the $y_1 y_2$-plane.

Let's look at some of them!

<u>First:</u> Suppose $B = 0$, $A > 0$

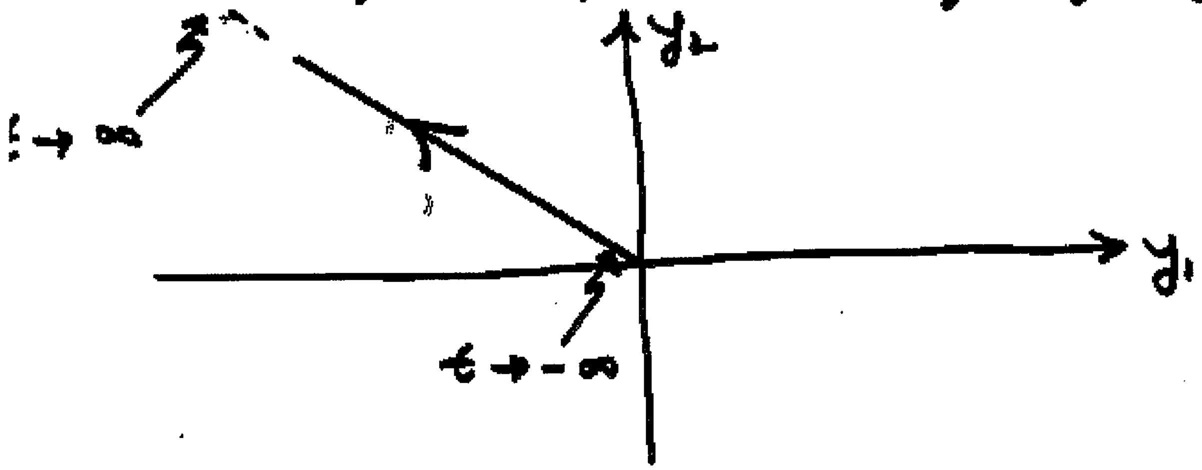
$$\Rightarrow \quad \underset{\sim}{y}(t) = A\begin{pmatrix}1\\-1\end{pmatrix}e^{t}$$

$$\Rightarrow \quad y_1(t) = Ae^{t} \ (>0), \quad y_2(t) = -Ae^{t} \ (<0)$$
$$= -y_1(t)$$



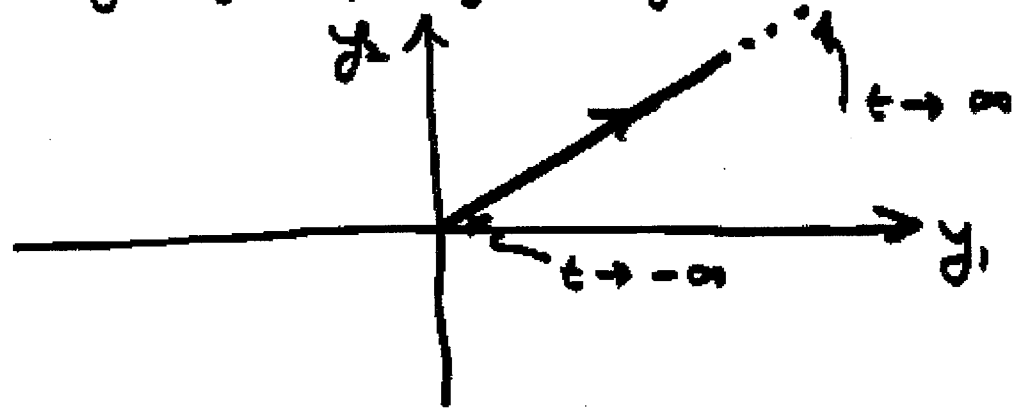
Choosing different (positive) A's corresponds to starting at different places on red line at $t = 0$.

For $B=0$, $A<0$, we have $y_1=-y_2$, $y_1<0$, $y_2>0$



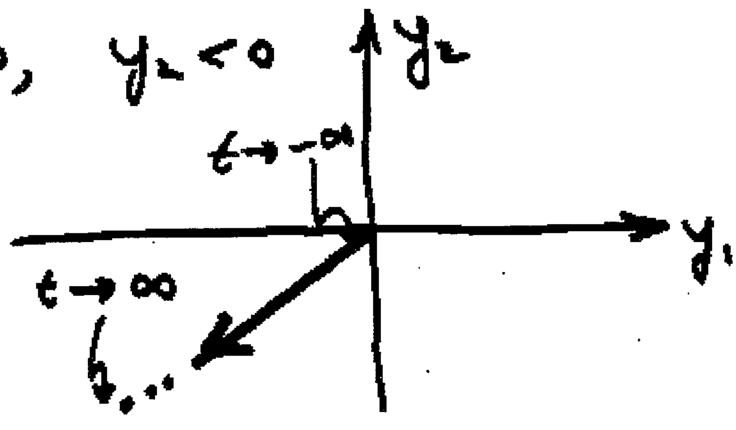
Similarly, for $A=0$, $B>0$ we get

$y_1=y_2$, $y_1>0$, $y_2>0$

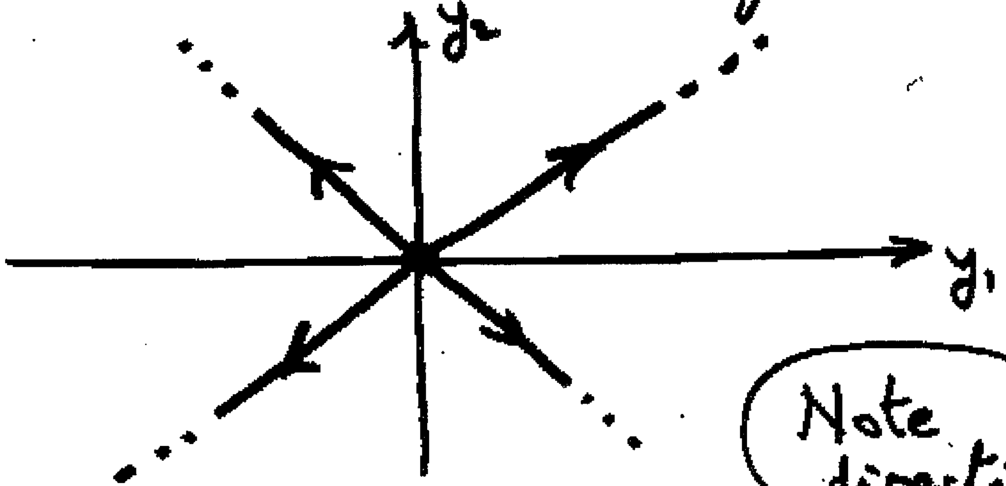


and for $A=0$, $B<0$ we get

$y_1=y_2$, $y_1<0$, $y_2<0$



Now we have so far 5 trajectories:-



Note directions

Note that they do not cross. They meet only at $t = -\infty$.

To get further info. about the phase-portrait, imagine a trajectory, and think of $y_2(y_1)$ on it. The <u>slope</u> of this curve is

$$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} \quad (\underline{\text{Chain rule}}\,!)$$

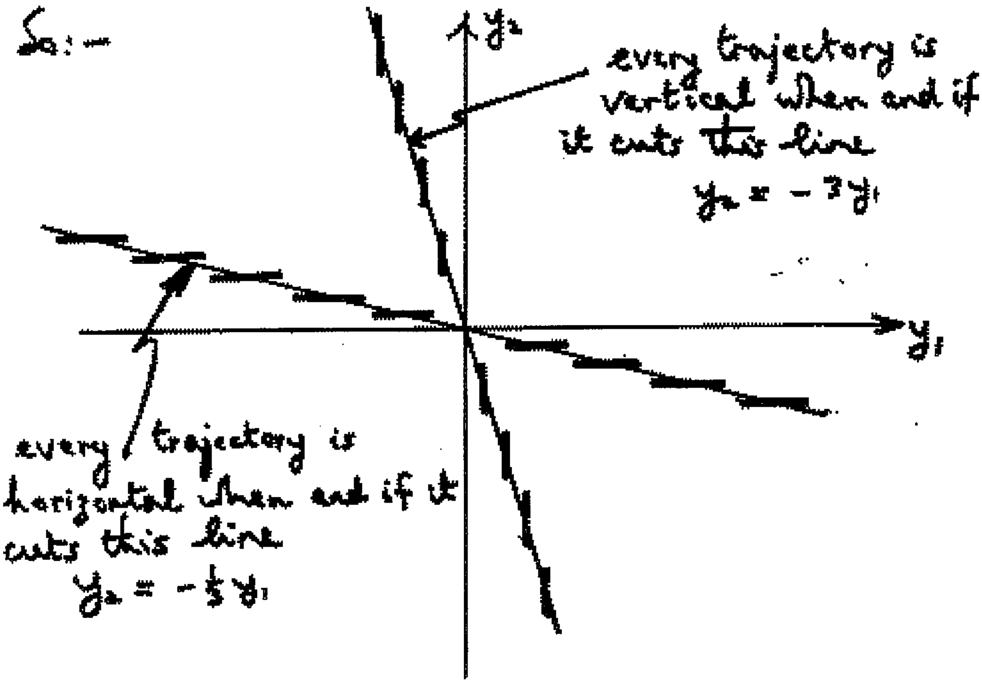
$$= \frac{\frac{1}{2}y_1 + \frac{3}{2}y_2}{\frac{3}{2}y_1 + \frac{1}{2}y_2} = \frac{y_1 + 3y_2}{3y_1 + y_2}$$

It follows that

$$\frac{dy_2}{dy_1} = 0 \text{ wherever } y_2 = -\tfrac{1}{3}y_1$$

$$\frac{dy_2}{dy_1} = \pm\infty \text{ wherever } y_2 = -3y_1$$

So:—



How find <u>directions</u> (of arrows)?

Need info. about $\frac{d\underset{\sim}{y}}{dt}$ ("velocity" vector)

– how things change with <u>time</u>.

So, look again at our system:–

$$\frac{dy_1}{dt} = \frac{3}{2}y_1 + \frac{1}{2}y_2$$

$$\frac{dy_2}{dt} = \frac{1}{2}y_1 + \frac{3}{2}y_2$$

Where $y_2 = -\frac{1}{3}y_1$, we have

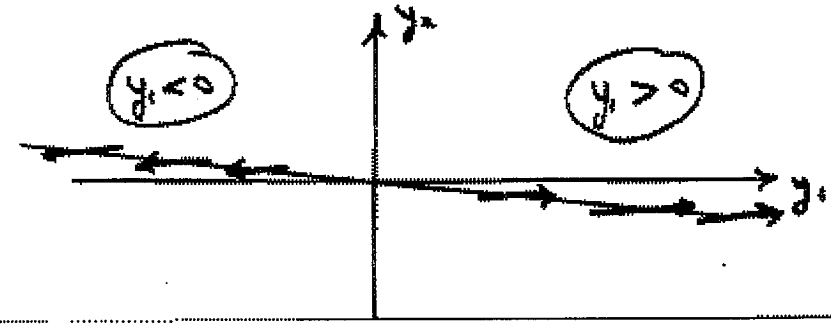
$$\frac{dy_1}{dt} = \frac{3}{2}y_1 + \frac{1}{2}\left(-\frac{1}{3}y_1\right) = \frac{4}{3}y_1$$

and so

$$\frac{dy_1}{dt} > 0 \text{ if } y_2 = -\frac{1}{3}y_1 \text{ \underline{and} } y_1 > 0$$

$$\frac{dy_1}{dt} < 0 \text{ if } y_2 = -\frac{1}{3}y_1 \text{ \underline{and} } y_1 < 0$$

So:



Similarly, where $y_2 = -3y_1$

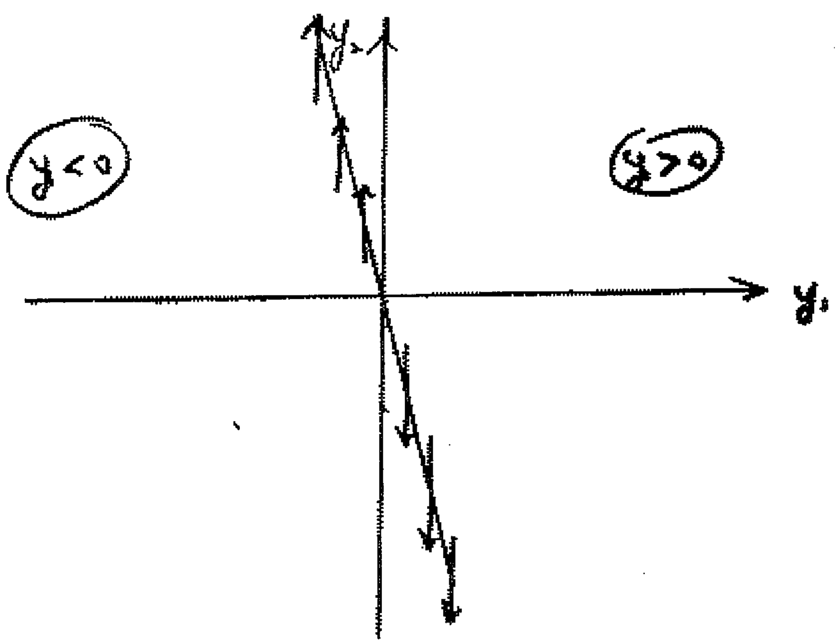
$$\frac{dy_2}{dt} = \frac{1}{2}y_1 + \frac{3}{2}(-3y_1) = -4y_1$$

and so

$$\frac{dy_2}{dt} < 0 \quad \text{if} \quad y_2 = -3y_1 \quad \underline{\text{and}} \quad y_1 > 0$$

$$\frac{dy_2}{dt} > 0 \quad \text{if} \quad y_2 = -3y_1 \quad \underline{\text{and}} \quad y_1 < 0$$

So:

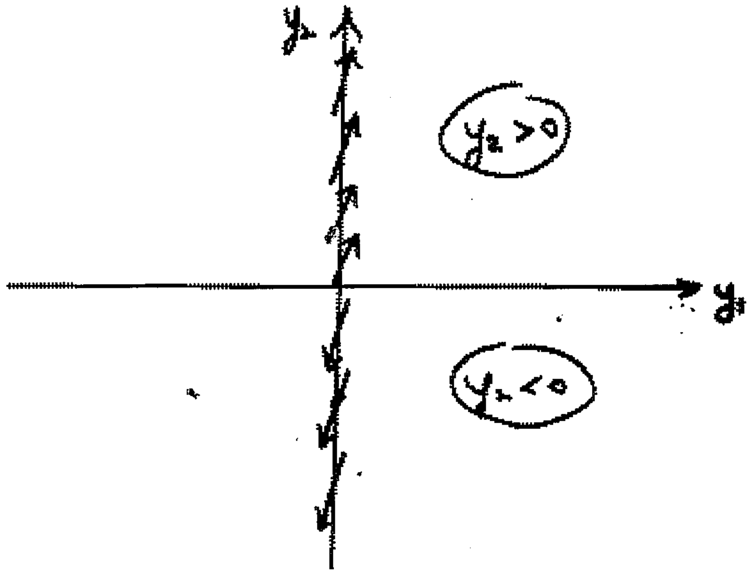


Can get further info. by considering trajectories where they cross other straight lines.

For example, where $y_1 = 0$, we have

$$\frac{dy_2}{dy_1} = \frac{y_1 + 3y_2}{3y_1 + y_2} = \frac{3y_2}{y_2} = 3 \quad (\text{for } y_2 \neq 0)$$

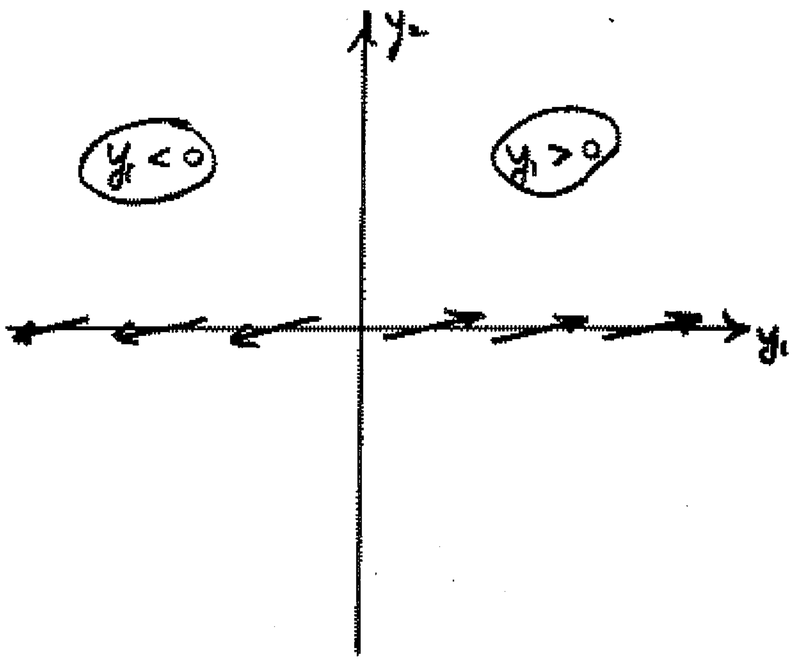
$$\frac{dy_1}{dt} = \tfrac{3}{2} y_1 + \tfrac{1}{2} y_2 = \tfrac{1}{2} y_2 \quad \begin{cases} > 0 \text{ if } y_2 > 0 \\ < 0 \text{ if } y_2 < 0 \end{cases}$$



Similarly, where $y_2 = 0$, we have

$$\frac{dy_2}{dy_1} = \frac{y_1 + 3y_2}{3y_1 + y_2} = \frac{y_1}{3y_1} = \frac{1}{3} \quad (\text{for } y_1 \neq 0)$$

$$\frac{dy_1}{dt} = \frac{3}{2} y_1 + \frac{1}{2} y_2 = \frac{3}{2} y_1 \begin{cases} > 0 & \text{if} \quad y_1 > 0 \\ < 0 & \text{if} \quad y_1 < 0 \end{cases}$$



Lecture summary:

1) Improper node: A has two real e'values with same sign.

2) $\underset{\sim}{y} = \underline{0}$ (origin) is a trajectory.

3) Know how to find 4 st. trajectories radiating from origin.

4) Know how to find $\frac{dy_2}{dy_1}$ and use to get info. about trajectories.

5) Know how to use $\frac{dy_1}{dt}$ (or $\frac{dy_2}{dt}$) to get info. about directions of arrows.

K pp. ~~142 - 144~~ 140 - 142