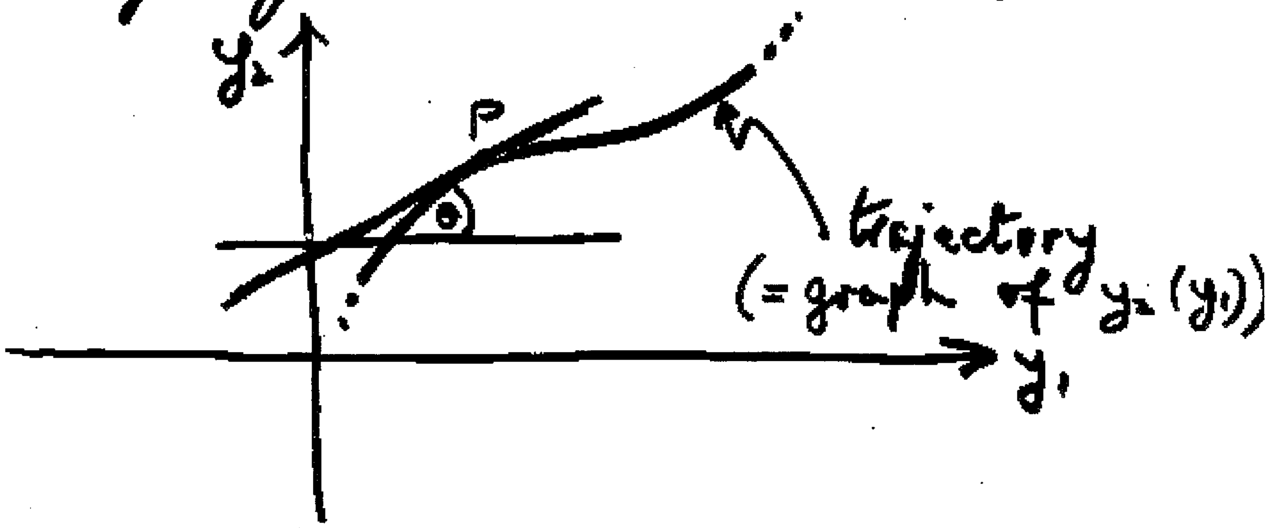
Lec. 6 MATH2100/2010

Expanding argument on bottom of p 5.5:



On trajectory (red), we have $(y_1(t), y_2(t))$ as coordinates of moving point P.
We can think of y_2 as a function of y_1 on the trajectory: $y_2(y_1)$ $[\Rightarrow y_2(t) = y_2(y_1(t))]$. The slope of the curve at the general point P ($= \tan\theta$) is given as usual by $\frac{dy_2}{dy_1}$.

By the chain rule, $\frac{dy_2}{dt} = \frac{dy_2}{dy_1}\frac{dy_1}{dt}$

$$\Rightarrow \frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} \quad \left(\frac{\text{bottom ODE}}{\text{top ODE}}\right)$$

Finally, we look at general solution again

$$\underset{\sim}{y}(t) = A\begin{pmatrix}1\\-1\end{pmatrix}e^{t} + B\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$$

and see that, as $t \to -\infty$, $\underset{\sim}{y}(t) \to \begin{pmatrix}0\\0\end{pmatrix}$, and also that, when t is large and negative,

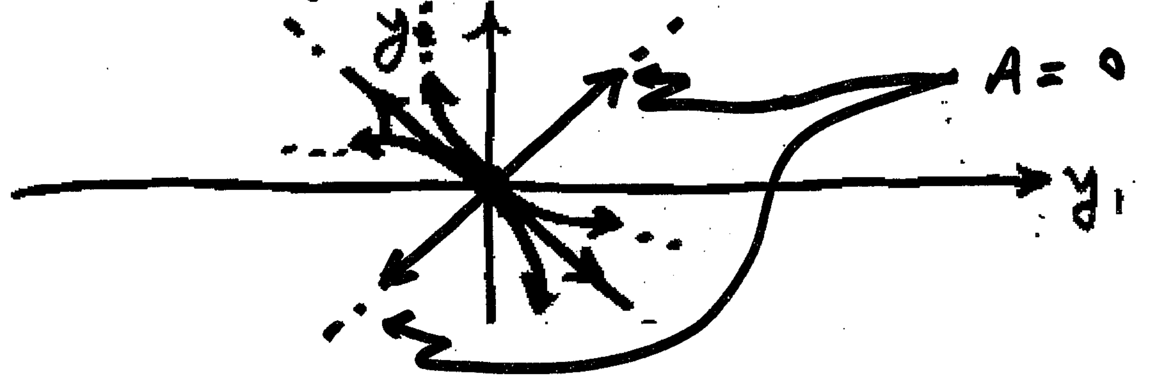
$$e^{2t} << e^{t}$$

So

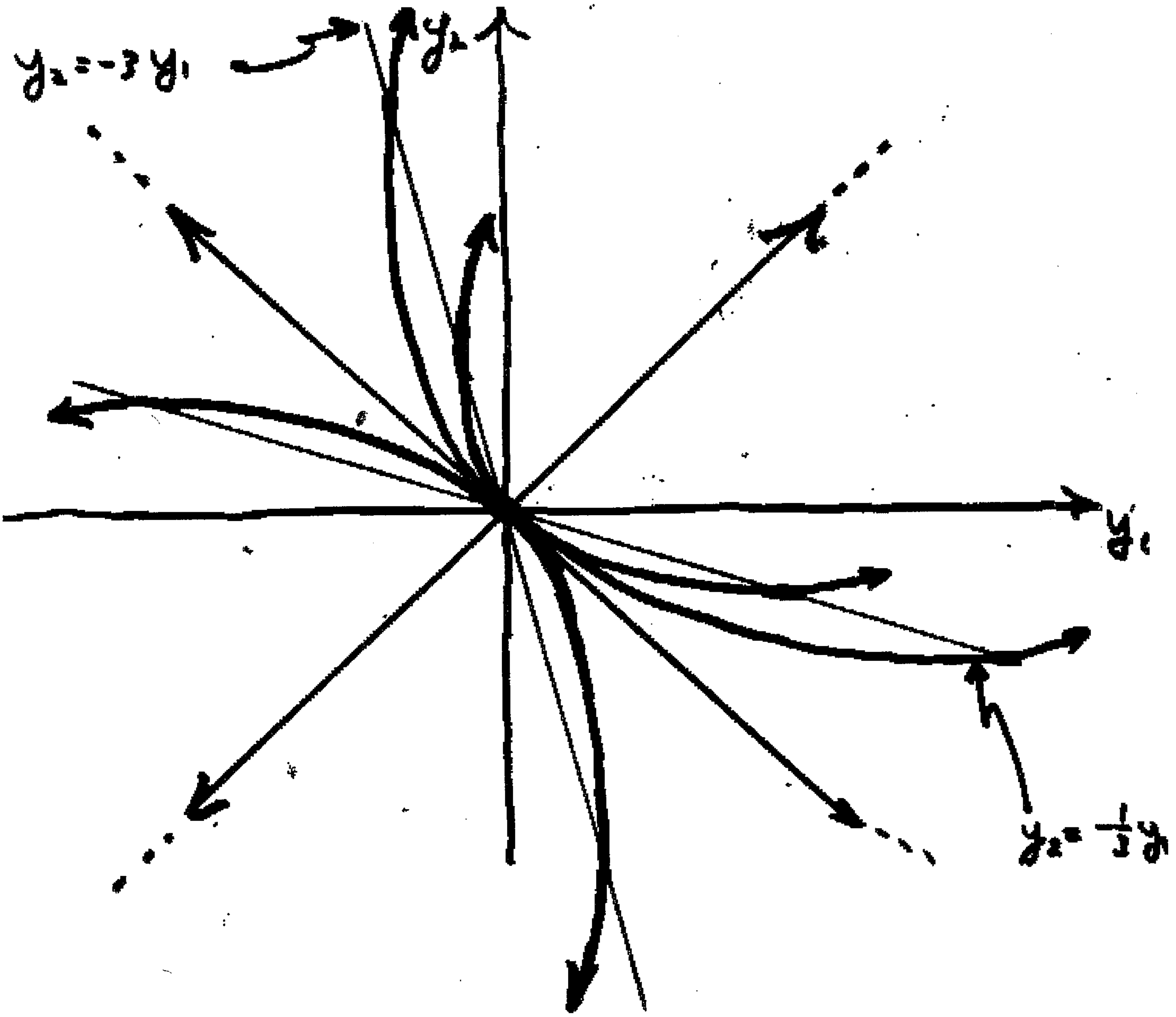
(if $A \neq 0$) $\quad \underset{\sim}{y}(t) \approx A\begin{pmatrix}1\\-1\end{pmatrix}e^{t}$

$$\Rightarrow \quad y_2(t) \approx -y_1(t)$$

From this we can see that all trajectories "<u>start</u>" at origin O at $t = -\infty$, and near O, we have:-



Now put whole thing together and get our *phase-portrait* for system (5.1):

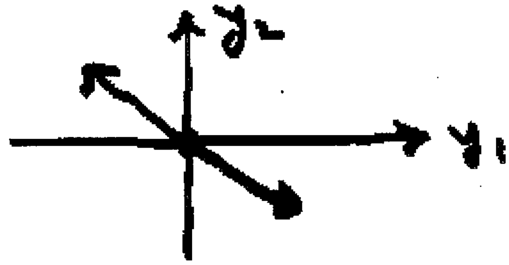


Improper *node* (repulsive)

and those along line $y_2 = y_1$

Note: 1) Every trajectory (except $\underline{y} = \underline{0}$) is tangential to $y_2 = -y_1$ ($y_1 > 0$ or $y_1 < 0$) as $t \to -\infty$ (i.e. at origin)

So: two limiting directions for all trajectories (except $\underline{y} = \underline{0}$ and those along $y_2 = y_1$) as $t \to -\infty$.



2) Trajectories never cross (they meet at $\underline{y} = \underline{0}$ as $t \to -\infty$).

$\underline{y}' = A\underline{y} \Rightarrow$ a definite (unique) direction for the one (unique) trajectory through any given point (except origin).

3) Can also have an <u>attractive</u> improper node, with all trajectories meeting at origin as $t \to +\infty$

This is the case where both e'values of A are real and negative.

(See K p. ~~[illegible]~~ 141 Ex. 1 Fig. ~~78~~ 81)

<u>Improper nodes characterized by:-</u>

$\begin{cases} \text{Two real e'values, same sign} \\ \text{(Only) two limiting directions near Origin O.} \end{cases}$

We say that origin O is a <u>critical</u> (or <u>equilibrium</u>) <u>point</u> of the system $\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t)$, or ~~that~~

$\underset{\sim}{y}(t) = \underset{\sim}{0}$ is a critical/equilibrium solution.

(For any matrix A).

(6.6)

TYPE 2 : Proper Node

Here $A = \mu I = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$ (μ = const.)

Then $A\underline{x} = \lambda \underline{x}$

$\Rightarrow \mu I \underline{x} = \lambda \underline{x}$

$\Rightarrow \mu \underline{x} = \lambda \underline{x}$

$\Rightarrow \lambda = \mu$ if $\underline{x} \neq \underline{0}$

See <u>every</u> nonzero $\underline{x}$ is an e'vector, with e'value μ.

Choose any two <u>linearly</u> <u>independent</u> e'vectors, such as

$$\underline{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \underline{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>General solution</u>:

$$\underline{y}(t) = A\,\underline{x}^{(1)} e^{\mu t} + B\,\underline{x}^{(2)} e^{\mu t}$$
$$= \begin{pmatrix} A \\ B \end{pmatrix} e^{\mu t}$$

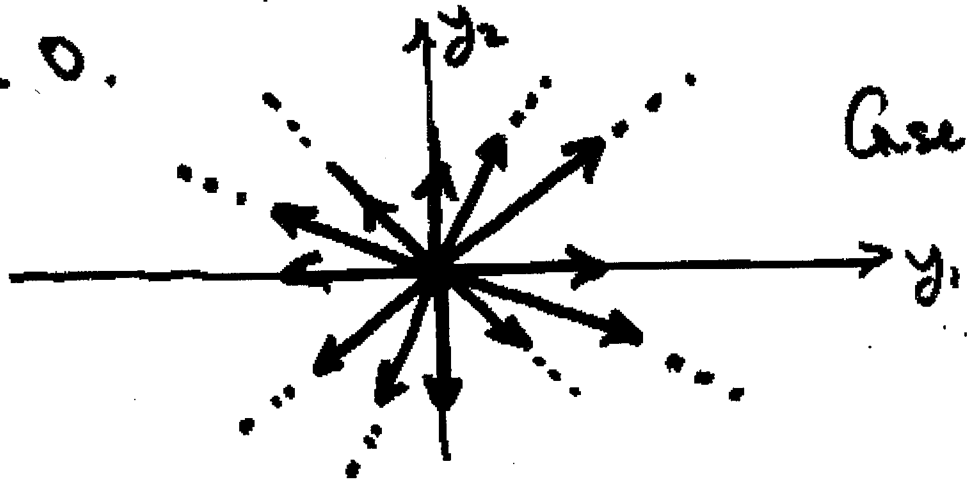
So $y_1(t) = Ae^{\mu t}$ $\qquad y_2(t) = Be^{\mu t}$

and $y_2 = \frac{B}{A} y_1$ (if $A \neq 0$)

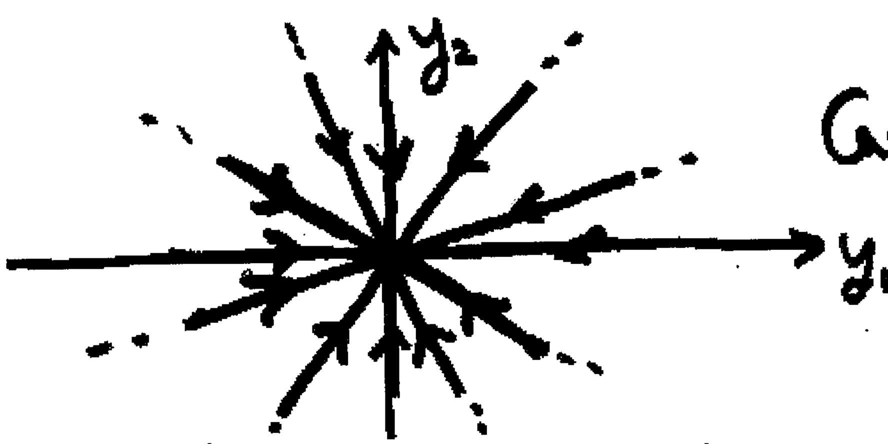
or $y_1 = \frac{A}{B} y_2$ (if $B \neq 0$)

or $\underset{\sim}{y} = \underset{\sim}{0}$ (if $A = 0 = B$)

See that (apart from critical solution $\underset{\sim}{y} = \underset{\sim}{0}$) all trajectories are st. lines through origin O.



Case $\mu > 0$



Case $\mu < 0$

<u>Proper node: Characterized by</u> trajectories in every possible direction at critical point.

TYPE 3: Saddle Point

Two real e'values of opposite sign.

Ex: $\underset{\sim}{y}'(t) = A\underset{\sim}{y}(t)$ $\qquad A = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix}$ (6.1)

E'value condition: $(\det(A - \lambda I) = 0)$

$$0 = \begin{vmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{vmatrix} = (7-\lambda)(-5-\lambda) + 32$$
$$= \lambda^2 - 2\lambda - 3$$
$$= (\lambda - 3)(\lambda + 1)$$

E'values are: $\lambda_1 = 3$, $\lambda_2 = -1$

Finding e'vectors: $\lambda_1 = 3$: $\qquad ((A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0})$

$$\begin{bmatrix} 7-3 & -8 \\ 4 & -5-3 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} 4u - 8v = 0 \\ 4u - 8v = 0 \end{array}\right\} \Rightarrow v = \tfrac{1}{2}u$$

choosing $u = 1$, we get $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$

$\lambda_2 = -1$:

$$\begin{bmatrix} 7+1 & -8 \\ 4 & -5+1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{array}{l} 8u - 8v = 0 \\ 4u - 4v = 0 \end{array}\right\} \Rightarrow \quad v = u$$

Choosing $u=1$, we get $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So, general solution of (6.1) is

$$\underset{\sim}{y}(t) = c_1 \underset{\sim}{x}^{(1)} e^{3t} + c_2 \underset{\sim}{x}^{(2)} e^{-t}$$

$$= c_1 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

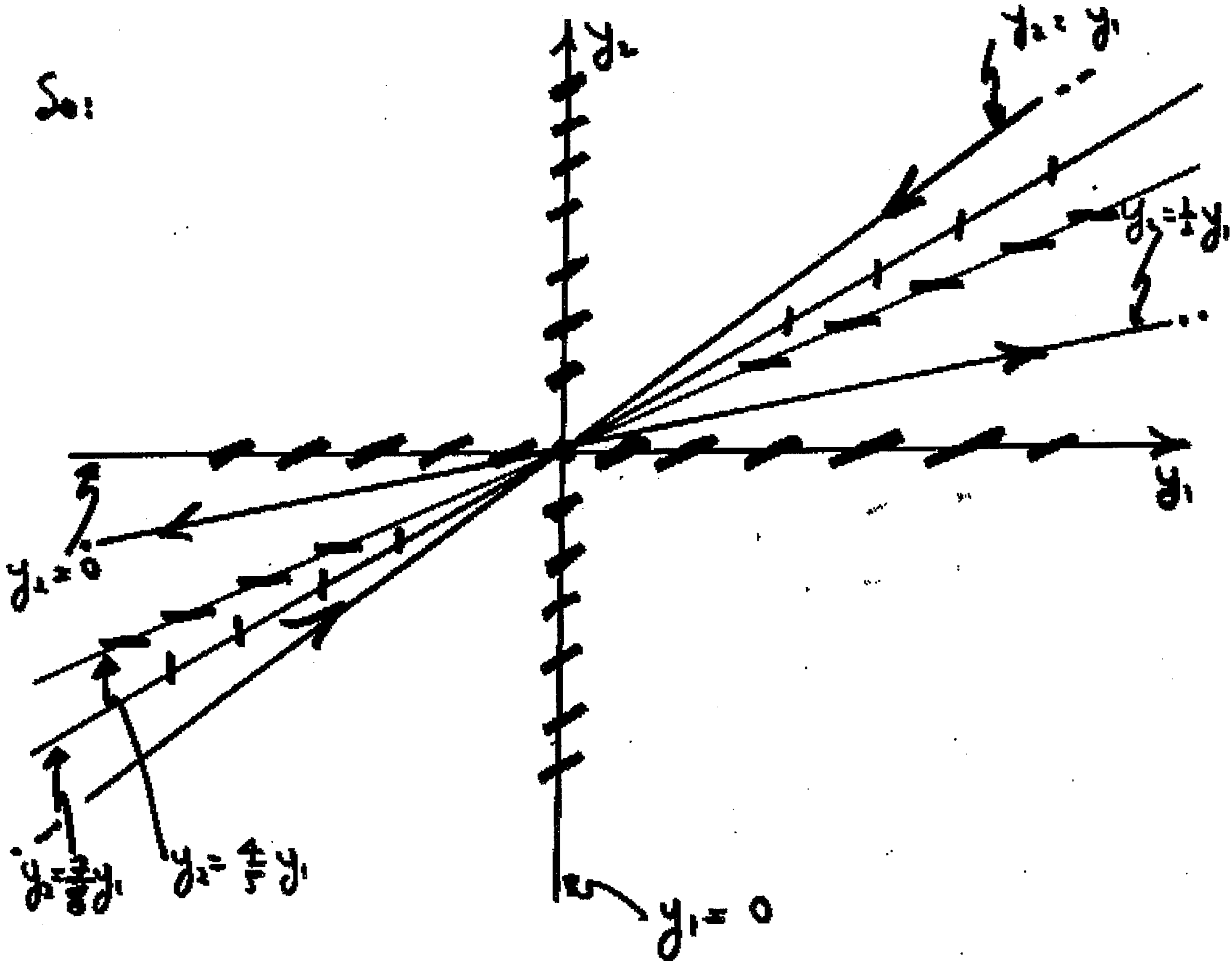
With $c_2 = 0$: two <u>outgoing</u> trajectories
(two halves of line $y_2 = \frac{1}{2} y_1$)

With $c_1 = 0$: two <u>incoming</u> trajectories
(two halves of line $y_2 = y_1$)

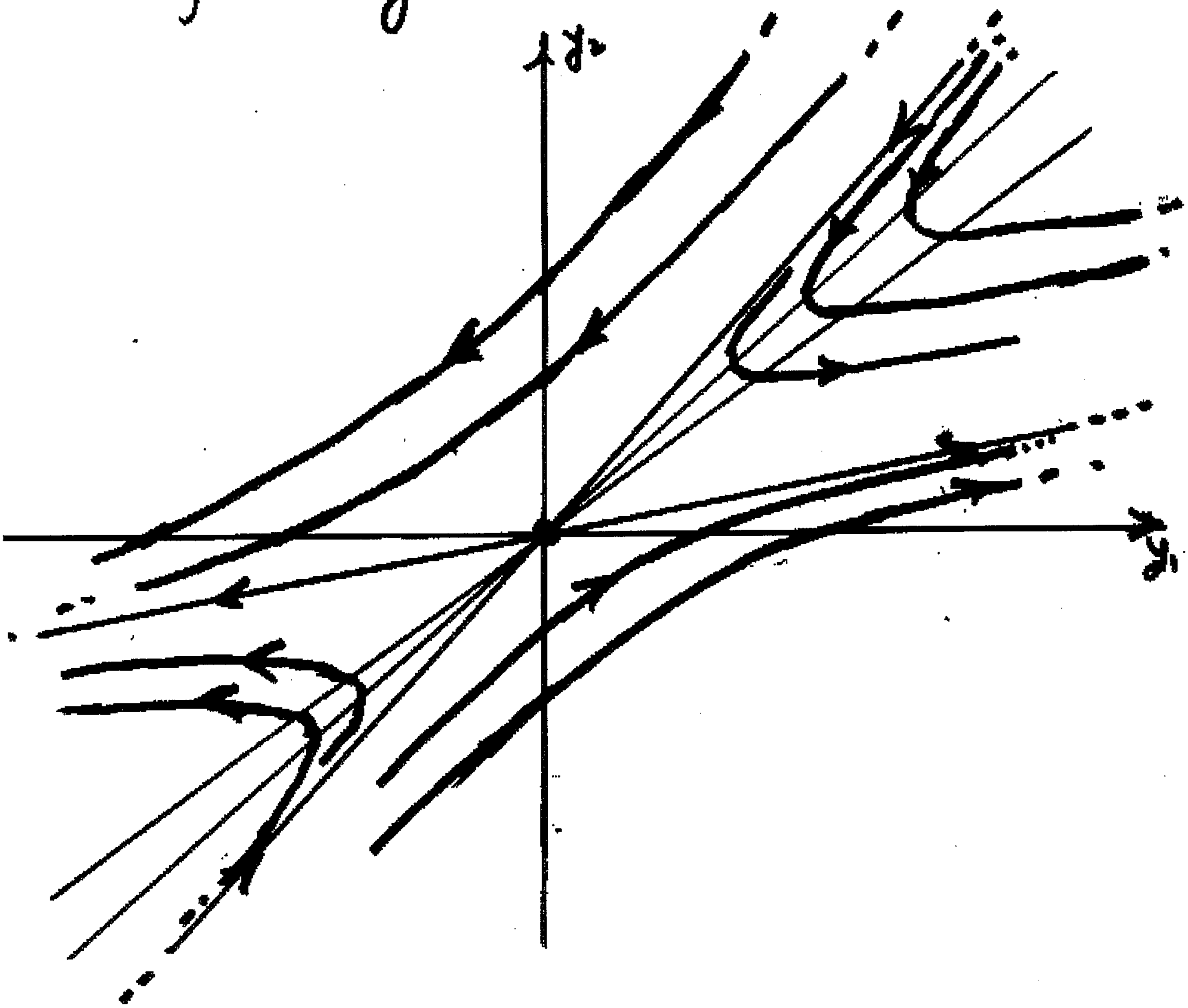
Also: $$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{4y_1 - 5y_2}{7y_1 - 8y_2}$$

$$= \begin{cases} 0 & \text{on line} \quad y_2 = \frac{4}{5}y_1 \\ \pm\infty & \text{on line} \quad y_2 = \frac{7}{8}y_1 \\ \frac{5}{8} & \text{on line} \quad y_1 = 0 \\ \frac{4}{7} & \text{on line} \quad y_2 = 0 \end{cases}$$

So:



Putting all together :-



For another example of a saddle, see
K p. ~~165~~ 143 Ex. 3 Fig. 8~~6~~3

Saddle point characterized by two real e'values of opposite sign. Get two incoming trajectories, + two outgoing trajectories. Rest by-pass origin O.

Summary:

1) Understand meaning and use of $\frac{dy_2}{dy_1}$ to help draw trajectories.

2) Understand how to find phase-portraits for improper node, proper node and saddle pt.

K pp 140-143 ~~162-165~~.