Lec. 8 MATH2100/2010

TYPE 5 Spiral (or Focus)

Eigenvalues form a complex-conjugate pair, with non-zero real part.
Trajectories spiral into or out of origin O.

EX: (8.1) $\underset{\sim}{y}'(t) = A\,\underset{\sim}{y}(t)$ $\qquad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

E'value condition: $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Eigenvectors: $\lambda_1 = 1+i$:

$$(A-\lambda_1 I)\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \begin{bmatrix} 1-1-i & 1 \\ -1 & 1-1-i \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} -iu + v &= 0 \\ -u - iv &= 0 \end{aligned}\right\} \Rightarrow v = iu$$

Choosing $u = 1$ we get $\underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda_2 = 1-i$:

Similarly ($i \leftrightarrow -i$), $\underset{\sim}{x}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

General solution: $\underset{\sim}{y}(t) = A\underset{\sim}{x}^{(1)}e^{(1+i)t} + B\underset{\sim}{x}^{(2)}e^{(1-i)t}$

$$= e^t\left\{A\begin{pmatrix} 1 \\ i \end{pmatrix}e^{it} + B\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{-it}\right\}$$

$$= e^t\left\{A\begin{pmatrix} 1 \\ i \end{pmatrix}[\cos t + i\sin t]\right.$$

$$\left. + B\begin{pmatrix} 1 \\ -i \end{pmatrix}[\cos t - i\sin t]\right\}$$

$$= e^t \left\{ \begin{pmatrix} (A+B)\cos t + i(A-B)\sin t \\ i(A-B)\cos t - (A+B)\sin t \end{pmatrix} \right\}$$

$\Rightarrow$

$$\underset{\sim}{y}(t) = e^t \left\{ C \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + D \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \right\}$$

Real form of gen. sol°.

$$C = A+B, \quad D = i(A-B) \quad \left[\Rightarrow \begin{array}{l} A = \frac{1}{2}(C - iD) \\ B = \frac{1}{2}(C + iD) \end{array} \right]$$

Getting path-equation (equation for trajectories):

Easiest from complex-form of gen solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+i)t} + B \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1-i)t}$$

$$\left. \begin{array}{l} y_1 = A e^{(1+i)t} + B e^{(1-i)t} \\ y_2 = iA e^{(1+i)t} - iB e^{(1-i)t} \end{array} \right\}$$

$\Rightarrow$

$$\left. \begin{array}{l} iy_1 + y_2 = 2iA e^{(1+i)t} \\ iy_1 - y_2 = 2iB e^{(1-i)t} \end{array} \right\}$$

$$\Rightarrow (iy_1 + y_2)(iy_1 - y_2) = -4AB e^{(1+i)t} e^{(1-i)t} = -4AB e^{2t}$$

$$\Rightarrow \quad -y_1^2 - y_2^2 = -4ABe^{2t}$$

$$= -4 \tfrac{1}{2}(C-iD)\tfrac{1}{2}(C+iD)e^{2t}$$

$$\Rightarrow \quad y_1^2 + y_2^2 = (C^2+D^2)e^{2t} \qquad (8.2)$$

spiral (outwards from origin)

["circle with continuously growing radius"]

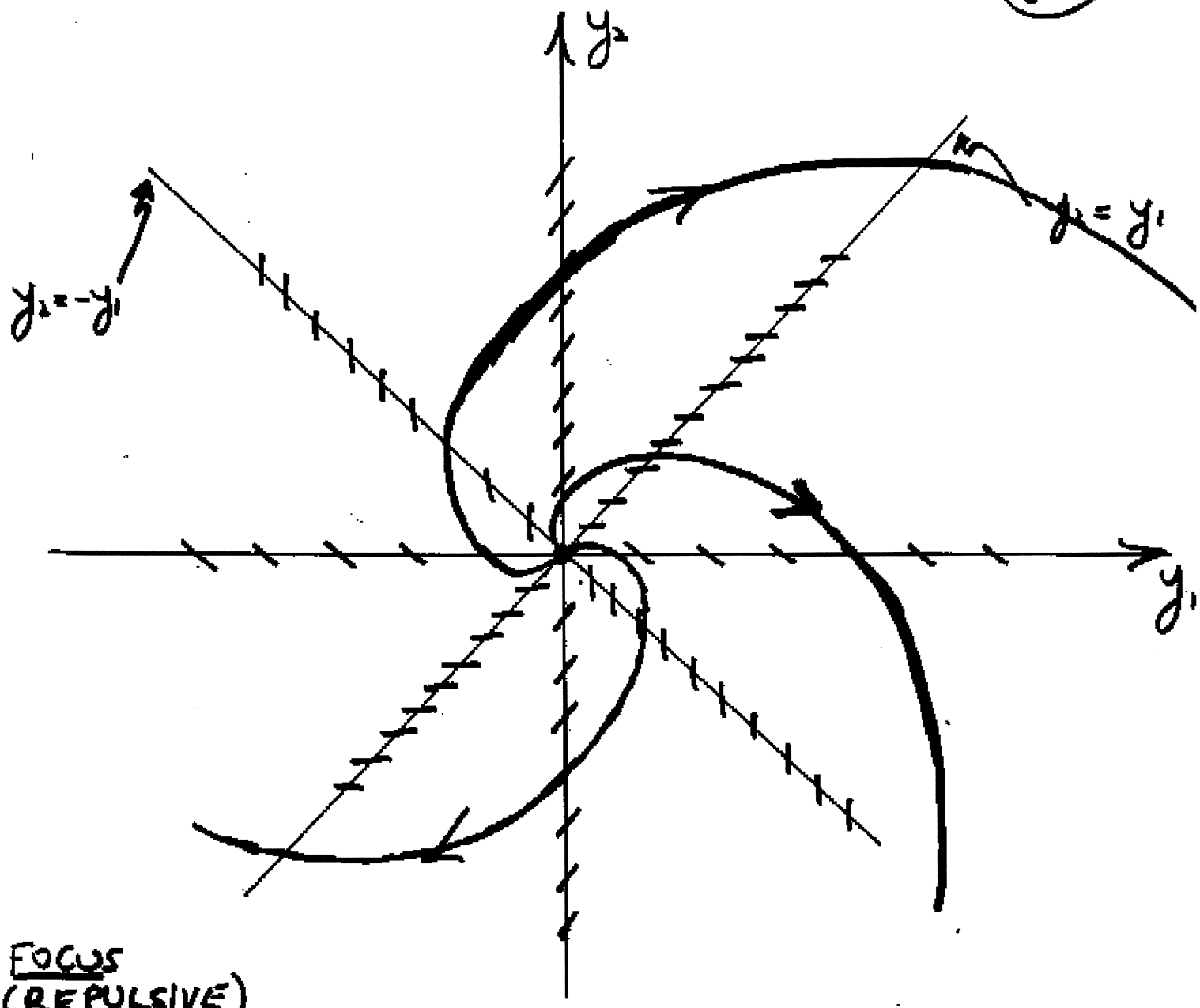
Trajectories spiral round and round origin

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Also: $\dfrac{dy_2}{dy_1} = \dfrac{dy_2/dt}{dy_1/dt} = \dfrac{-y_1+y_2}{y_1+y_2}$

$$= \begin{cases} 0 & \text{where} \quad y_2 = y_1 \\ \pm\infty & \text{where} \quad y_2 = -y_1 \\ 1 & \text{where} \quad y_1 = 0 \\ -1 & \text{where} \quad y_2 = 0 \end{cases}$$

8.5



FOCUS
(REPULSIVE)

See directions from (8.2) ($e^{2t} \to \infty$ as $t \to \infty$)

OR Use one of ODEs (8.1), say $y_1'(t) = y_1(t) + y_2(t)$

$\Rightarrow y_1'(t)$ positive when $y_1 > 0$, and $y_2 > 0$

$\Rightarrow y_1$ increasing in first quadrant

For another example of a focus (or spiral)
See K p 166 Ex. 5 Fig. 82

TYPE 6 : Degenerate Node

Only one eigenvalue, and only one linearly-independent eigenvector.

[Don't confuse with proper node, where have $A = \mu I$: one e'value μ, but two linearly-independent e'vectors.]

EX: (8.1) $\underset{\sim}{y}'(t) = A\,\underset{\sim}{y}(t)$ $\qquad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

Eigenvalue condition : $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)^2 - 0 = 0$$

$$\Rightarrow (\lambda+1)^2 = 0$$

$$\Rightarrow \lambda_1 = -1$$ one e'value only!

(8.7)

E' vectors: $\lambda_1 = -1$:

$$(A - \lambda_1 I)\underline{x} = \underline{0} \Rightarrow \begin{bmatrix} -1+1 & 1 \\ 0 & -1+1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} v = 0 \\ 0 = 0 \end{matrix}\right\} \Rightarrow \quad v = 0$$

Choosing $u = 1$, we get $\underline{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Now we have one solution $\underline{y}(t) = \underline{x}^{(1)} e^{-t}$.

- how can we find another?

Trick that works is to try to find another sol^n in form:

$$\underline{y}(t) = t\,\underline{x}^{(1)} e^{\lambda_1 t} + \underline{u}\, e^{\lambda_1 t}$$

and look for constant $\underline{u}$ such that $\underline{y}$ is a solution of (8.3)

[See K p. 167 for discussion.]

So in our example, we try

$$\underset{\sim}{y}(t) = t\underset{\sim}{x}^{(1)} e^{-t} + \underset{\sim}{u} e^{-t} \qquad (8.4)$$

$$\Rightarrow \quad \underset{\sim}{y}'(t) = \underset{\sim}{x}^{(1)} e^{-t} - t\underset{\sim}{x}^{(1)} e^{-t} - \underset{\sim}{u} e^{-t} \qquad (8.5)$$

$$A\underset{\sim}{y} = tA\underset{\sim}{x}^{(1)} e^{-t} + A\underset{\sim}{u} e^{-t}$$

$$= -t\underset{\sim}{x}^{(1)} e^{-t} + A\underset{\sim}{u} e^{-t} \qquad (8.6)$$

(using $A\underset{\sim}{x}^{(1)} = -\underset{\sim}{x}^{(1)}$)

Subbing (8.5) and (8.6) in (8.3) ($\underset{\sim}{y}' = A\underset{\sim}{y}$)

we get:

$$\underset{\sim}{x}^{(1)} e^{-t} - t\underset{\sim}{x}^{(1)} e^{-t} - \underset{\sim}{u} e^{-t} = -t\underset{\sim}{x}^{(1)} e^{-t} + A\underset{\sim}{u} e^{-t}$$

$$\Rightarrow \quad (A\underset{\sim}{u} + \underset{\sim}{u}) e^{-t} = \underset{\sim}{x}^{(1)} e^{-t}$$

$$(A - (-1)I)\underset{\sim}{u} = \underset{\sim}{x}^{(1)}$$

[In general case, get $(A - \lambda_1 I)\underset{\sim}{u} = \underset{\sim}{x}^{(1)}$ at this point.]

Setting $\underset{\sim}{u} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, and noting that

$$A + 1 \cdot I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \underset{\sim}{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we have $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \left.\begin{matrix} \beta = 1 \\ 0 = 0 \end{matrix}\right\} \Rightarrow \beta = 1$$

Choosing $\alpha = 0$, we get $\underset{\sim}{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Now $\underset{\sim}{y}(t) = \underset{\sim}{x}^{(1)} e^{-t}$ and $\underset{\sim}{y}(t) = t\underset{\sim}{x}^{(1)} e^{-t} + \underset{\sim}{u} e^{-t}$

are each solutions of system (8.3).

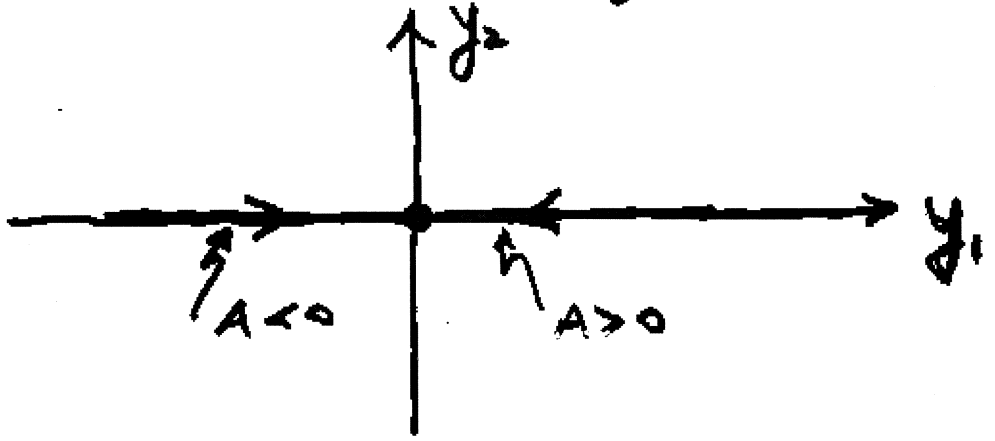
General solution:

$$\underset{\sim}{y}(t) = A\underset{\sim}{x}^{(1)} e^{-t} + B\left[t\underset{\sim}{x}^{(1)} e^{-t} + \underset{\sim}{u} e^{-t}\right]$$

Now $t\underset{\sim}{x}^{(1)} + \underset{\sim}{u} = \begin{pmatrix} t \\ 1 \end{pmatrix}$, so we have

$$\underset{\sim}{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t} + B\begin{pmatrix} t \\ 1 \end{pmatrix}e^{-t}$$

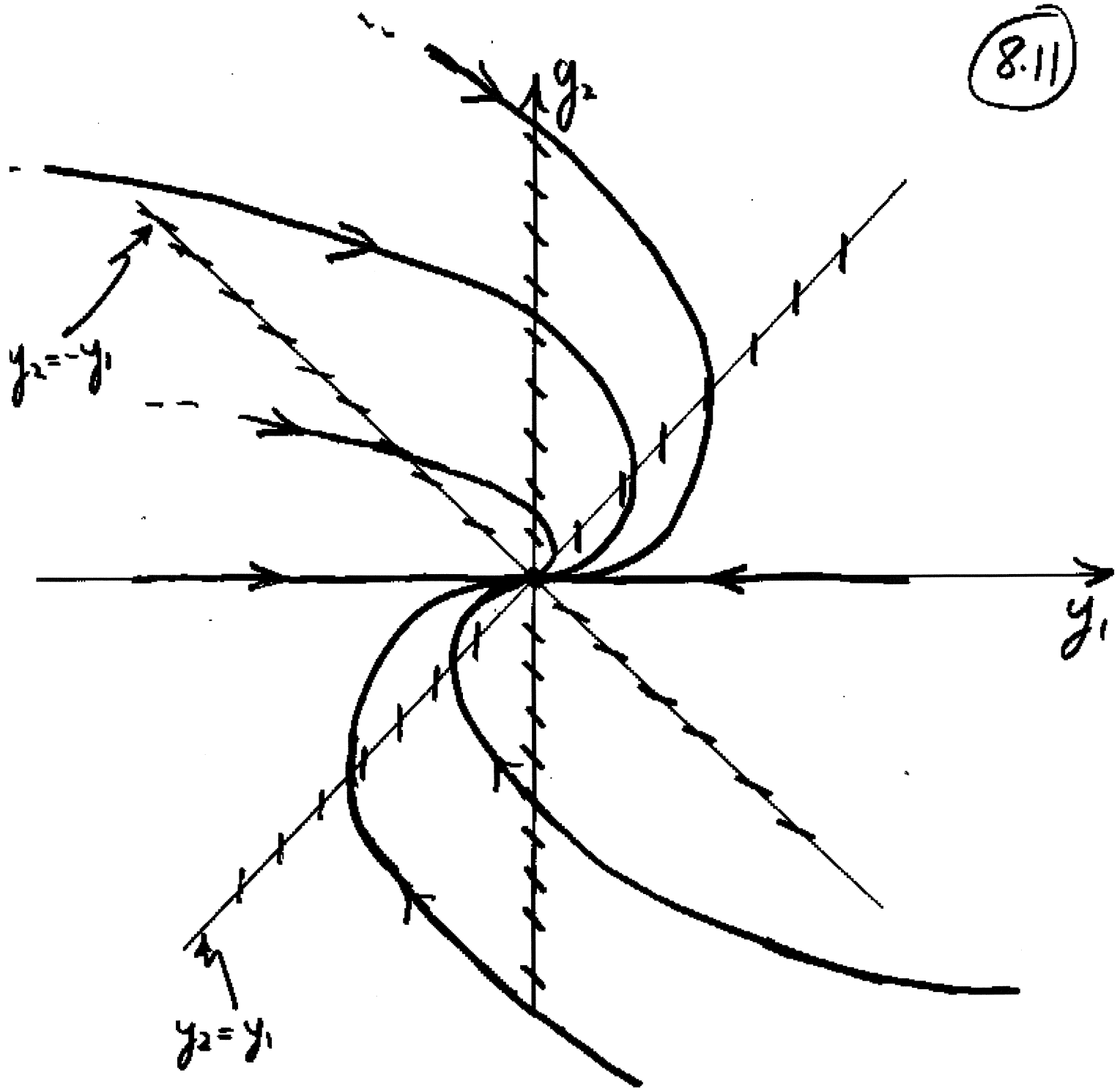
Trajectories? $B=0 \Rightarrow y_1 = Ae^{-t}$, $y_2 = 0$



$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Also: $\dfrac{dy_2}{dy_1} = \dfrac{dy_2/dt}{dy_1/dt} = \dfrac{-y_2}{-y_1+y_2}$

$$= \begin{cases} 0 & \text{where } y_2 = 0 \\ \pm\infty & \text{where } y_2 = y_1 \\ -1 & \text{where } y_1 = 0 \\ -\frac{1}{2} & \text{where } y_2 = -y_1 \end{cases}$$



Degenerate node. (attractive)

For another example, see

K p. 168 Ex. 6 Fig. 83

(repulsive case)

Summary:

1) Understand when get focus (or spiral) and when get degenerate node.

2) Be able to get general solution (and general solution in real form) for focus, and be able to derive path-equation

3) Look carefully at derivation of general solution for degenerate node.

4) Be able to plot trajectories in both cases.

5) Look carefully at differences between phase-portraits for improper node, proper node, focus and degenerate node.

K § 3.7.