Take the following example.

\[ A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \] where \( \lambda_1 = 1, \lambda_2 = -1 \) and \( x^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( x^2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \).

So that

\[ X = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \] \(\text{det}X = 2\) and \( X^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \)

Then if you work out

\[ X^{-1}AX = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Which is a diagonal matrix with the eigenvalues of \( A \) as its diagonal entries!

**Using Diagonalisation to Solve Linear Systems of ODE's**

Now suppose that \( A \) were the matrix for a linear system of first order ODE's.

\[ \dot{y} = Ay \] for example \( \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \)

If \( A \) has two distinct real eigenvalues it is diagonalisable and you can change to new variables \((z_1, z_2)\), whose equations of motion have a particularly simple form.

The new variables \((z_1, z_2)\) are given by \( y = Xz \).

Substituting these into \( \dot{y} = Ay \) gives

\[ \frac{dXz}{dt} = AXz \quad \text{or} \quad X\dot{z} = AXz \]

Now multiplying both sides by \( X^{-1} \) gives \( \dot{z} = X^{-1}AXz = Dz \).

In other words in the \( z \) variables we have

\[ \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \implies \dot{z}_i = \lambda_iz_i \implies z_i = c_ie^{\lambda_it} \]

The \( z_1 \) equation decouples from the \( z_2 \) and so is easily solved.

We usually call this **Normal Form**.

Systems in **Normal Form** also have simpler phase portraits as the eigenvectors of the matrix are \( x^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( x^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), so the axes in the Phase Portrait are the straight line trajectories of the system.

For the example given above

\[ \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \implies z_1 = c_1e^t, \ z_2 = c_2e^{-t} \]

Transforming back \( y = Xz \) gives \( y_1 = c_1e^t + c_2e^{-t} \) and \( y_2 = c_1e^t + 3c_2e^{-t} \).
Relevance to Phase Curves.

ANY system with **two real distinct eigenvalues** is a linear transformation away from

\[
\begin{pmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{pmatrix} = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
\]

Since the eigenvectors of a system such as this, in Normal Form, are parallel to the axes, the axes are trajectories.

Also the phase curves are easy to solve for because

\[
\frac{dz_2}{dz_1} = \frac{\lambda_2 z_2}{\lambda_1 z_1} \implies z_2 = C z_1^{\frac{\lambda_2}{\lambda_1}}
\]

(Can you show this?)

The system in the \(y_i\) variables is a linear transformation, \(y = Xz\), away from Normal form.

Now a linear transformation can sheer, rotate and enlarge or reduce, so in a general system with two real distinct eigenvalues the phase curves will be sheered, rotated and or enlarged or reduced versions of these curves.

**Complex Eigenvalues**

There is a similar result for the case of complex eigenvalues. In this case the matrix of transformations \(X\) is comprised of the real and imaginary parts of the eigenvectors and, using this, you can show that ANY system with complex eigenvalues is a linear transformation away from one whose phase curves are ellipses or logarithmic spirals.

So in a general system with complex eigenvalues the logarithmic spiral may be sheered, rotated and enlarged or reduced.